J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 12, Number 4 (November 2005), Pages 275–287

DISCRETE TORSION AND NUMERICAL DIFFERENTIATION OF BINORMAL VECTOR FIELD OF A SPACE CURVE

MYUNGJIN JEON

ABSTRACT. Geometric invariants are basic tools for geometric processing and computer vision. In this paper, we give a linear approximation for the differentiation of the binormal vector field of a space curve by using the forward and backward differences of discrete binormal vectors. Two kind of discrete torsion, say, backward torsion τ_b and forward torsion τ_f can be defined by the dot product of the (backward and forward) discrete differentiation of binormal vectors that are linear approximations of torsion. Using Frenet formula and Taylor series expansion, we give error estimations for the discrete torsions. We also give numerical tests for a curve. Notably the average of τ_b and τ_f looks more stable in errors.

1. INTRODUCTION

For space curves in the 3-dimensional Euclidean space E^3, the curvature and torsion characterize a curve up to isometry (*cf.* do Carmo [5]). So these are very important geometric invariant in computer vision and geometric modeling. Furthermore if we have Frenet frame with curvature and torsion, then we can reconstruct the given space curve as it is. In this context, geometric invariants are frequently used in many applications of geometric processing and computer vision. So the accurate estimation of geometric invariants of curves from its discrete approximations are important. In many cases of computer graphics and computer vision, the space curve is given as a polygon which is an approximation of the original curve. The problem is how can we define the geometric invariants such as curvature and torsion, tangent vector, principal normal vector, binormal vector that approximate the original invariants.

Received by the editors September 15, 2005 and, in revised form, September 30, 2005.
2000 *Mathematics Subject Classification.* 65D25, 68U05, 65D18, 68R99, 52C99.
Key words and phrases. torsion estimation, numerical differentiation of vector field, space curve, Frenet formula.

For the curvature approximation, many authors has been studied(Anoshkina, Belyaev & Seidel [1], Borrelli, Cazals & Morvan [3], Coeurjolly, Serge & Laure [7], Costa [8], Langer, Belyaev & Seidel [14] and Maltret & Daniel [17]). But the torsion of a space curve is not yet sufficiently studied. Only some discrete definitions are proposed. Roughly speaking, this is because the curvature is a second derivative and the torsion is the third.

Let γ be a space curve. Let $-h_2 < -h_1 < 0 < k_1 < k_2$ and

$$P_{-2} = \gamma(-h_2), P_{-1} = \gamma(-h_1), P_0 = \gamma(0), P_1 = \gamma(k_1), P_2 = \gamma(k_2). \tag{1}$$

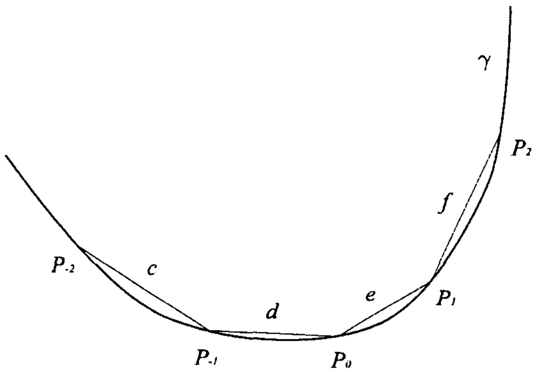

Figure 1. curve γ and polygon $\{P_{-2}, P_{-1}, P_0, P_1, P_2\}$

In Boutin [4], the discrete torsion at P_i is defined by the height of the tetrahedron formed by the vertices $\{P_{-1}, P_0, P_1, P_2\}$ measured from P_2 with normalization. In Langer, Belyaev & Seidel [14], the torsion is approximated by the angle between the discrete binormal vectors b_i and b_1 that are defined by the unit normal vector of the planes determined by the vertices P_{-1}, P_0, P_1 and P_0, P_1, P_2. But the torsion is originally defined by the derivative of the binormal vector of a curve. So the approximation of the derivative of the binormal vector field is a crucial step in finding the discrete torsion of a space curve.

In this paper, we present linear approximations of the derivative of the binormal vector of a space curve. From this approximation, we define a discrete torsion that approximates the torsion of a curve with errors of order 1.

2. Approximation of the Derivative of Binormal Vector Field of a Space Curve

In this section we approximate the derivative of the binormal vector field of a space curve.

Let $\mathcal{P} = \{P_{-2}, P_{-1}, P_0, P_1, P_2\}$ be a polygon in E^3, then the polygon can be regarded as an approximation of a smooth curve γ interpolating the polygon. Assume the curve γ is parameterized by arclength, then the Frenet apparatus $\mathbf{t}, \mathbf{n}, \mathbf{b}, \kappa, \tau$ at $P_i = \gamma(0)$ are given by

$$\begin{aligned} \mathbf{t} &= \gamma'(0) && (\textit{tangent vector}) \\ \mathbf{n} &= \frac{\gamma''(0)}{\|\gamma''(0)\|} && ((\textit{principal})\ \textit{normal vector}) \\ \mathbf{b} &= \mathbf{t} \times \mathbf{n} && (\textit{binormal vector}) \\ \kappa &= \|\gamma''(0)\| && (\textit{curvature}) \\ \tau &= < \mathbf{b}', \mathbf{n} > && (\textit{torsion}) \end{aligned}$$

where $\gamma'(0)$ and $\gamma''(0)$ are the first and second derivative of the curve γ, $\mathbf{b}'$ is the derivative of the binormal vector field of γ at $\gamma(0)$, $\|\cdot\|$ denotes the norm of vector, $< \mathbf{b}', \mathbf{n} >$ is the inner product of $\mathbf{b}'$, $\mathbf{n}$ and $\times$ denotes the cross product. The Frenet formula is

$$\begin{aligned} \mathbf{t}' &= \kappa \mathbf{n} \\ \mathbf{n}' &= -\kappa \mathbf{t} - \tau \mathbf{b} \\ \mathbf{b}' &= \tau \mathbf{n} \end{aligned}$$

where $\mathbf{t}', \mathbf{n}', \mathbf{b}'$ are the derivatives of tangent, normal and binormal vectors at $\gamma(0)$ respectively. (Here we follow the convention of do Carmo [5] for the sign of torsion.)

The unit tangent vector $\gamma'(0)$ can be approximated by the following vector with errors of order 2 for $i = -1, 0, 1$. (see Jeon [9] and Langer, Belyaev & Seidel [14]).

$$T_i = \frac{||\Delta P_i||}{(||\nabla P_i|| + ||\Delta P_i||)} \cdot \frac{\nabla P_i}{||\nabla P_i||} + \frac{||\nabla P_i||}{(||\nabla P_i|| + ||\Delta P_i||)} \cdot \frac{\Delta P_i}{||\Delta P_i||} \tag{2}$$

where ∇ and Δ denote the backward and forward differences, respectively.

$$\nabla P_i = P_i - P_{i-1}, \ \Delta P_i = P_{i+1} - P_i. \tag{3}$$

The binormal vector of a curve is a unit normal vector to the osculating plane. Since the plane defined by the three point P_{i-1}, P_i, P_{i+1} converges to the osculating plane of the curve γ at P_i as $P_{i-1} \to P_i, P_{i+1} \to P_i$, the binormal and principal (resp.) normal vectors can be approximated by the followings for $i = -1, 0, 1$:

$$B_i = \frac{\nabla P_i \times \Delta P_i}{||\nabla P_i \times \Delta P_i||} \quad \text{and} \quad N_i = B_i \times T_i \tag{4}$$

In many practical situations, curves are given by discrete data such as polygons so that we need to describe the geometric invariants by the discrete point and edge lengths. For convenience, put

$$c = ||P_{-1} - P_{-2}||,\ d = ||P_0 - P_{-1}||,\ e = ||P_1 - P_0||,\ f = ||P_2 - P_1||$$

as in Figure 1. According to Langer, Belyaev & Seidel [14], the tangent vector and binormal vectors can be approximated using the Taylor series expansion as follows.

$$\begin{aligned}
T_0 =& \mathbf{t}\left(1 - \frac{de}{8}\kappa^2 + \frac{d^2e - de^2}{12}\kappa\kappa' + O(d,e)^4\right) \\
&+ \mathbf{n}\left(\frac{de}{6}\kappa' - \frac{d^2e - de^2}{24}(\kappa'' - \kappa\tau^2) + O(d,e)^4\right) \\
&+ \mathbf{b}\left(-\frac{de}{6}\kappa\tau + \frac{d^2e - de^2}{24}(2\kappa'\tau + \kappa\tau') + O(d,e)^4\right) \\
B_{-1} =& \mathbf{t}\left(-\frac{cd + d^2}{6}\kappa\tau + O(c,d)^3\right) \\
&+ \mathbf{n}\left(-\frac{c + 2d}{3}\tau + \frac{c^2 + 3cd + 3d^2}{12}\tau' + \frac{c^2 + cd + d^2}{18}\frac{\kappa'}{\kappa}\tau + O(c,d)^3\right) \\
&+ \mathbf{b}\left(1 - \frac{c^2 + 4cd + 4d^2}{18}\tau^2 + O(c,d)^3\right) \\
B_0 =& \mathbf{t}\left(\frac{de}{6}\kappa\tau + O(d,e)^3\right) \\
&+ \mathbf{n}\left(\frac{e - d}{3}\tau + \frac{d^2e - de + e^2}{12}\tau' + \frac{d^2 + de + e^2}{18}\frac{\kappa'}{\kappa}\tau + O(d,e)^3\right) \\
&+ \mathbf{b}\left(1 - \frac{(d - e)^2}{18}\tau^2 + O(d,e)^3\right) \\
B_1 =& \mathbf{t}\left(-\frac{fe + e^2}{6}\kappa\tau + O(e,f)^3\right) \\
&+ \mathbf{n}\left(\frac{f + 2e}{3}\tau + \frac{f^2 + 3fe + 3e^2}{12}\tau' + \frac{f^2 + fe + e^2}{18}\frac{\kappa'}{\kappa}\tau + O(e,f)^3\right) \\
&+ \mathbf{b}\left(1 - \frac{f^2 + 4fe + 4e^2}{18}\tau^2 + O(e,f)^3\right)
\end{aligned}$$

where $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are the original tangent, normal and binormal vectors of the curve γ at P_0, respectively; $(\cdot)'$, $(\cdot)''$ denote the first and second derivatives of functions at $\gamma(0)$; and $O(x)^k$ denotes the k-th order of errors with respect to x.

Several kinds of technique have been developed on the numerical differentiation. These can be classified into five categories; finite difference type, polynomial interpolation type, operator type, lozenge diagrams, undetermined coefficients(cf. Chapra & Canale [6], Khan & Ohba [10, 11, 12], Knowles & Wallace [13] and Li [16]). We use the finite difference to obtain an approximation of the derivative of binormal vector field.

Conventionally the divided difference is given by the finite difference(forward or backward difference) divided by the edge length between two points.

For example, for the polygon given by equation (1), the backward and forward divided differences of the discrete binormal vector field B_{-1}, B_0, B_1 at P_0 are defined by

$$\frac{B_0 - B_{-1}}{d}, \ \frac{B_1 - B_0}{e}.$$

But the differences of discrete binormal vectors are related to three edges so that we define a discrete derivative of the binormal vectors at P_0 using the forward and backward differences of discrete binormal vectors and three edge lengths.

Definition 2.1. The backward, forward and average discrete derivatives of the binormal vector are defined by

$$B_0^b = \frac{3(B_0 - B_{-1})}{c+d+e}$$
$$B_0^f = \frac{3(B_1 - B_0)}{d+e+f}$$
$$B_0^a = \frac{1}{2}(B_0^b + B_0^f)$$

Using Taylor expansion, we can find an error estimation for B^b, B^f and B^a with the derivative $\mathbf{b}'$ of binormal vector field.

Theorem 2.1. *The discrete derivatives B^b, B^f and B^a defined above are linear approximations of the derivative $\mathbf{b}'$ of (original) binormal vector field.*

Proof. By direct computations, we have

$$
\begin{aligned}
B_0^b =& \mathbf{t}\left(\frac{d}{2}\kappa\tau + O(c,d,e)^2\right) \\
&+ \mathbf{n}\left(\tau + \frac{e-c-2d}{4}\tau' + \frac{(e-c)}{6}\frac{\kappa'}{\kappa}\tau + O(c,d,e)^2\right) \\
&+ \mathbf{b}\left(\frac{c+3d-e}{6}\tau^2 + O(c,d,e)^2\right) \\
B_0^f =& \mathbf{t}\left(-\frac{e}{2}\kappa\tau + O(d,e,f)^2\right) \\
&+ \mathbf{n}\left(\tau + \frac{f-d+2e}{4}\tau' + \frac{(f-d)}{6}\frac{\kappa'}{\kappa}\tau + O(d,e,f)^2\right) \\
&+ \mathbf{b}\left(\frac{d-3e-f}{6}\tau^2 + O(d,e,f)^2\right) \\
B_0^a =& \mathbf{t}\left(\frac{d-e}{4}\kappa\tau + O(c,d,e,f)^2\right) \\
&+ \mathbf{n}\left(\tau + \frac{(f-c+e-d)}{12}\frac{\kappa'}{\kappa}\tau + \frac{(f-c)+3(e-d)}{8}\tau' + O(c,d,e,f)^2\right) \\
&+ \mathbf{b}\left(\frac{c-f+4(d-e)}{12}\tau^2 + O(c,d,e,f)^2\right)
\end{aligned}
$$

Since $\mathbf{b}' = \tau\mathbf{n}$, B_0^f, B_0^b and B_0^a are linear approximations of $\mathbf{b}'$. □

Note that the average derivative shows a symmetric feature. In case $c = d = e = f$, the linear terms in the average derivative B_0^a vanishes.

So in this case B_0^a is a second order approximation of the derivative $\mathbf{b}'$.

3. Discrete Torsion

The torsion is defined as the rate of change of the binormal vector field in the direction of principal normal vector, say, $\tau = \mathbf{b}\cdot\mathbf{n}$.(see Belyaev [2] and do Carmo [5]) So we have to estimate the discrete normal vector N_i.

Theorem 3.1. *The discrete normal vector N_0 is estimated by the following.*

$$
\begin{aligned}
N_0 =& \mathbf{t}\left(-\frac{de}{6}\kappa' + O(d,e)^2\right) \\
&+ \mathbf{n}\left(1 - \frac{de}{8}\kappa^2 - \frac{(d-e)^2}{18}\tau^2 + O(d,e)^2\right) \\
&+ \mathbf{b}\left(\frac{d-e}{3}\tau - \frac{d^2-de+e^2}{12}\tau' - \frac{d62+de+e^2}{18}\frac{\kappa'}{\kappa}\tau + O(d,e)^2\right)
\end{aligned}
\tag{5}
$$

Proof. Direct computation using Frenet frame yields the expression (5).

The approximation T_0 for the tangent vector is of second order, but this theorem means that N_0 is a first order approximation for normal vector. This is because B_0 is only a first order approximation for binormal vector $\mathbf{b}$. However if $d = e$ then the approximation is of second order. As a result, we have a first order approximation of the torsion.

Theorem 3.2. *Define backward, forward and average discrete torsions by*

$$\tau_b = B_0^b \cdot N_0 \tag{6}$$

$$\tau_f = B_0^f \cdot N_0 \tag{7}$$

$$\tau_a = \frac{1}{2}(\tau_b + \tau_f) \tag{8}$$

then τ_b, τ_f and τ_a are linear approximations of the torsion τ.

Proof. Direct computation yields

$$
\begin{aligned}
\tau_b &= \tau + \frac{e-c-2d}{4}\tau' + \frac{(e-c)}{6}\frac{\kappa'}{\kappa}\tau + O(c,d,e)^2 \\
\tau_f &= \tau + \frac{2e+f-d}{4}\tau' + \frac{(f-d)}{6}\frac{\kappa'}{\kappa}\tau + O(d,e,f)^2 \\
\tau_a &= \tau + \frac{(f-c+e-d)}{12}\frac{\kappa'}{\kappa}\tau + \frac{f-c+3(e-d)}{8}\tau' + O(c,d,e,f)^2
\end{aligned}
$$

Hence τ_b and τ_f are linear approximations for the torsion.

As the average derivative B^a, the average discrete torsion τ_a shows a symmetric feature. If all edge lengths are same, i.e. $c = d = e = f$ then the linear terms of τ_a vanishes.

In this case we have a second order approximation of the torsion.

4. Experimental Results

In this section, we will compare the discrete torsions τ_b, τ_f with real torsion for Clelia curve. Clelia curve is defined by the curve on a sphere with constant $\frac{\phi}{\theta}$ where ϕ and θ are the longitude and colatitude(the angular distance from a pole) (cf. [18]). For $\frac{\phi}{\theta} = 1$, the Clelia curve is parameterized by

$$\gamma(t) = (\sin(t)\cos(t), \sin(t)\sin(t), \cos(t)).$$

We have used Maple 9.5 for computations to test the accuracy of the approximation of torsions.

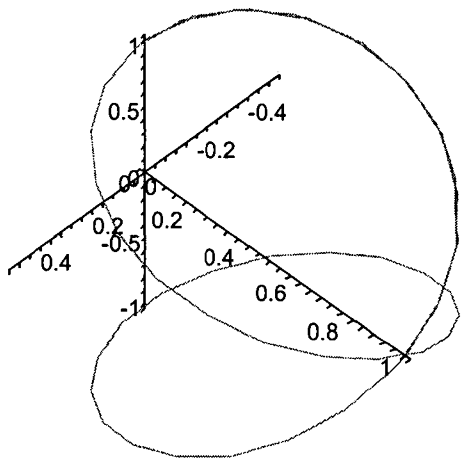

Figure 2. Clelia curve for $\frac{\phi}{\theta} = 1$

Table 1 is the result at the point $P_0 = \gamma(\frac{6}{5})$. The torsion of γ at P_0 is $\tau_0 = 0.7352311234$. P_{-2}, P_{-1}, P_1, P_2 are defined by

$$P_{-2} = \gamma(t_0 + a_{-2}h), \quad P_{-1} = \gamma(t_0 + a_{-1}h) \tag{9}$$

$$P_1 = \gamma(t_0 + a_1 h), \qquad P_2 = \gamma(t_0 + a_2 h) \tag{10}$$

where $t_0 = 6/5$, $h = 2^{-i}(i = 1, \ldots, 9)$ and $a_{-2} = -3, a_{-1} - 2, a_1 = 1, a_2 = 4$.

Table 1: Test for Clelia curve at $t = 6/5$, $\tau_0 = 0.7352311234$

h	τ_b	τ_f	ϵ_b	ϵ_f
1/2	0.5050585959	0.7256198368	0.2301725275	0.0096112866
1/4	0.6662899597	0.7466573200	0.0689411637	-0.0114261966
1/8	0.7125017907	0.7397827422	0.0227293327	-0.0045516188
1/16	0.7267530943	0.7369507112	0.0084780291	-0.0017195878
1/32	0.7317071721	0.7359352237	0.0035239513	-0.0007041003
1/64	0.7336505365	0.7355403745	0.0015805869	-0.0003092511
1/128	0.7345171353	0.7353486705	0.0007139881	-0.0001175471
1/256	0.7348921978	0.7353642673	0.0003389256	-0.0001331439
1/512	0.7342568226	0.7352693939	0.0009743008	-0.0000382705

Now we compute torsions at various points on γ. Table 2 and Table 3 are the discrete torsions and errors for randomly chosen polygon $\mathcal{P} = \{P_1, \ldots, P_{15}\}$ on γ with 15 vertices.

The average of backward and forward torsion, say $\tau_m = \frac{\tau_b + \tau_f}{2}$, seems more stable which is a linear approximation of the torsion. We also test for randomly chosen 23 vertices.

Table 2: Test for Clelia curve for approximating polygon consisting of 15 vertices (torsions)

i	τ	τ_b	τ_f	τ_m
1	0.0000000000	-0.1545159372	0.2042471103	0.0248655866
2	0.5451347114	0.2156853869	0.4944279506	0.3550566687
3	0.7377742302	0.5442338094	0.6769883035	0.6106110565
4	0.7071190254	0.6910401129	0.7140623560	0.7025512344
5	0.0617851337	0.7104811262	0.7132090612	0.7118450937
6	-0.5734683856	0.7042182100	0.6648351881	0.6845266990
7	-0.7412633040	0.6234114241	0.4167896813	0.5201005526
8	-0.7372233558	0.3594184339	0.0021032369	0.1807608355
9	-0.2950875693	0.0021441952	-0.3909892653	-0.1944225350
10	-0.6137402982	-0.4339878462	-0.6069407547	-0.5204643005
11	-0.7341640572	-0.6642185856	-0.7240157046	-0.6941171451
12	-0.7495060512	-0.7338109740	-0.7267949780	-0.7303029760
13	-0.7321838442	-0.6993138302	-0.6497442551	-0.6745290427
14	-0.6113476374	-0.6289597233	-0.5091031439	-0.5690314336
15	-0.4225438450	-0.4924178070	-0.1655000669	-0.3289589370

Table 3: Test for Clelia curve for approximating polygon consisting of 15 vertices (errors)

i	ϵ_b	ϵ_f	ϵ_m
1	-0.1545159372	0.2042471103	0.0248655866
2	-0.2373247689	0.0414177948	-0.0979534871
3	-0.1277592624	0.0049952317	-0.0613820153
4	-0.0369659805	-0.0139437374	-0.0254548590
5	-0.0395172172	-0.0367892822	-0.0381532497
6	-0.0227863454	-0.0621693673	-0.0424778564
7	-0.0048058481	-0.2114275909	-0.1081167196
8	0.0977521398	-0.2595630572	-0.0809054586
9	0.2972317645	-0.0959016960	0.1006650343
10	0.1797524520	0.0067995435	0.0932759977
11	0.0699454716	0.0101483526	0.0400469121
12	0.0156950772	0.0227110732	0.0192030752
13	0.0328700140	0.0824395891	0.0576548015
14	-0.0176120859	0.1022444935	0.0423162038
15	-0.0698739620	0.2570437781	0.0935849080

Table 4: Test for Clelia curve for approximating polygon consisting of 23 vertices (torsions)

i	τ	τ_b	τ_f	τ_m
1	0.0000000000	-0.1453536775	0.1582897195	0.0064680210
2	0.2631686012	0.1572127404	0.3949378323	0.2760752864
3	0.5572189120	0.4065478130	0.5861343218	0.4963410674
4	0.6740480928	0.6055999354	0.6844177304	0.6450088329
5	0.7268113620	0.6914784953	0.7231588203	0.7073186578
6	0.7478175156	0.7314583806	0.7435990889	0.7375287347
7	0.7497338454	0.7440344108	0.7346480994	0.7393412551
8	0.7367596740	0.7308124762	0.7202214010	0.7255169386
9	0.7074953784	0.7113410189	0.6392875006	0.6753142597
10	0.5870007947	0.6212693065	0.4927518711	0.5570105888
11	0.4228915159	0.4800423592	0.2140279182	0.3470351387
12	0.0402763431	0.2065109027	-0.0708166303	0.0678471363
13	-0.2826166674	-0.0732440473	-0.3666046154	-0.2199243314
14	-0.4690623621	-0.3708831461	-0.5384305696	-0.4546568578
15	-0.6428655504	-0.5494152635	-0.6591167378	-0.6042660007
16	-0.7053588444	-0.6689231141	-0.7096063637	-0.6892647388
17	-0.7393336656	-0.7120419967	-0.7323751379	-0.7222085674

18	-0.7498756890	-0.7361926196	-0.7413338435	-0.7387632316
19	-0.7457766690	-0.7389540397	-0.7229121987	-0.7309331192
20	-0.7173323610	-0.7150584019	-0.6751128187	-0.6950856104
21	-0.6519338226	-0.6681953439	0.3949378323	-0.6219391938
22	-0.5187775903	-0.5629065558	-0.3894416802	-0.4761741180
23	-0.2708050373	-0.3821068535	-0.1453453792	-0.2637261164

Table 5: Test for Clelia curve for approximating polygon consisting of 23 vertices (errors)

i	ϵ_b	ϵ_f	ϵ_m
1	-0.1453536775	0.1582897195	0.0064680210
2	-0.1059558608	0.1317692311	0.0129066852
3	-0.1506710990	0.0289154098	-0.0608778446
4	-0.0684481574	0.0103696376	-0.0290392599
5	-0.0353328667	-0.0036525417	-0.0194927042
6	-0.0163591350	-0.0042184267	-0.0102887809
7	-0.0056994346	-0.0150857460	-0.0103925903
8	-0.0059471978	-0.0165382730	-0.0112427354
9	0.0038456405	-0.0682078778	-0.0321811187
10	0.0342685118	-0.0942489236	-0.0299902059
11	0.0571508433	-0.2088635977	-0.0758563772
12	0.1662345596	-0.1110929733	0.0275707932
13	0.2093726201	-0.0839879480	0.0626923360
14	0.0981792160	-0.0693682075	0.0144055043
15	0.0934502869	-0.0162511874	0.0385995497
16	0.0364357303	-0.0042475193	0.0160941056
17	0.0272916689	0.0069585277	0.0171250982
18	0.0136830694	0.0085418455	0.0111124574
19	0.0068226293	0.0228644703	0.0148435498
20	0.0022739591	0.0422195423	0.0222467506
21	-0.0162615213	0.0762507791	0.0299946288
22	-0.0441289655	0.1293359101	0.0426034723
23	-0.1113018162	0.1254596581	0.0070789209

5. Conclusion

In this paper, we proposed a discrete differentiation of the binormal vector field of a discrete curve(a polygon approximating a smooth curve) which is a first order approximation of the original differentiation of binormal vectors. Using the discrete differentiation of binormal vector field, we find a first order approximation of the torsion of the curve in a natural way. This has an advantage on the computation costs comparing with the torsion estimator proposed in Lewiner, Gomes Jr., Lopes & Craizer [15]. At least we don't need to solve a system of linear equations.

In this paper we find discrete torsions based on the discrete differentiation of (discrete) binormal vector fields. So we can expect that the discrete curvature can be defined in the same way and this can lead to the Frenet formula in the discrete situations.

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DEPARTMENT OF COMPUTER AIDED MATHEMATICAL INFORMATION SCIENCE, SEMYUNG UNIVERSITY, SAN 21-1, SINWOUL-DONG, JECHON, CHUNGBUK 390-711, KOREA
Email address: `mjjeon@semyung.ac.kr`