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COMMON FIXED POINTS WITHOUT CONTINUITY IN FUZZY METRIC SPACES

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ABSTRACT. The aim of this paper is to prove some common fixed point theorems for six discontinuous mappings in non complete fuzzy metric spaces with condition of weak compatibility.

1. INTRODUCTION AND PRELIMINARIES

The concept of fuzzy sets was introduced initially by Zadeh [28] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [5], Erceg [6], Kaleva & Seikkala [17], Kramosil & Michalek [18] have introduced the concept of fuzzy metric spaces in different ways. Recently many authors have also studied the fixed point theory in these fuzzy metric spaces ([1], [2], [7], [9], [10], [11], [12], [19], [23], [24], [25], [26], [27]).

Grabiec [9] followed Kramosil & Michalek [18] and obtained the fuzzy version of Banach's fixed point theorem.

In 1976, Jungck [13] established common fixed point theorems for commuting maps generalizing the Banach's fixed point theorem. Sessa [22] defined a generalization of commutativity, which is called *weak commutativity.* Further Jungck [14] introduced more generalized commutativity, so called *compatibility.* Mishra, Sharma & Singh [19] introduced the concept of compatibility in fuzzy metric spaces. In 1998, Jungck & Rhoades [16] introduced the notion of weakly compatible maps and showed that compatible maps are weakly compatible but converse need not true.

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Sharma & Deshpande [26, 27] improved the results of Mishra, Sharma & Singh [19], Cho [3], Cho, Pathak, Kang & Jung [4], Sharma [24] and Sharma & Deshpande [25]. They proved common fixed point theorems for weakly compatible maps in fuzzy metric spaces without taking any mapping continuous.

In this paper, we prove some common fixed point theorems for six mappings by taking a different contractive type condition for class of weakly compatible maps in non complete fuzzy metric spaces, without taking any continuous mapping. We improve and extend the results of Mishra, Sharma & Singh [19], Cho [3], Cho, Pathak, Kang & Jung [4] and Sharma & Deshpande [26]. We also improve the results of Sharma [24] and Sharma & Deshpande [25, 27].

Definition 1.1 (Schweizer & Sklar [21]). A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is called a *continuous t-norm* if $([0, 1], *)$ is an Abelian topological monoid with the unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Examples of t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.2 (Kramosil & Michalek [18]). The 3-tuple $(X, M, *)$ is called a *fuzzy metric space* (shortly *FM-space*) if X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set in $X^2 \times [0, \infty]$ satisfying the following conditions for, all $x, y, z \in X$ and $t,\ s > 0$,

(FM-1) $M(x, y, 0) = 0$,
(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$, and
(FM-5) $M(x, y, \bullet) : [0,1] \to [0,1]$ is left continuous.

In what follows, $(X, M, *)$ will denote a fuzzy metric space. Note that $M(x, y, t)$ can be thought as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with ∞ and we can find some topological properties and examples of fuzzy metric spaces in George & Veeramani [8].

In the following example, we know that every metric induces a fuzzy metric.

Example 1 (George & Veeramani [8]). Let (X, d) be a metric space. Define $a*b = ab$ or $a * b = \min\{a, b\}$ and, for all $x, y \in X$ and $t > 0$, let

$$M(x, y, t) = \frac{t}{t + d(x, y)}. \tag{1.1}$$

Then $(X, M, *)$ is a fuzzy metric space. We call the fuzzy metric M induced by the metric d the *standard fuzzy metric.* On the other hand, note that there exists no metric on X satisfying equation (1.1).

Lemma 1.1 (Grabiec [9]). *For all $x, y \in X, M(x, y, \bullet)$ is nondecreasing.*

Definition 1.3 (Grabiec [9]). Let $(X, M, *)$ is a fuzzy metic space:

(a) A sequence $\{x_n\}$ in X is said to be *convergent to a point* $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$), if

$$\lim_{n\to\infty} M(x_n, x, t) = 1,$$

for all $t > 0$.

(b) A sequence $\{x_n\}$ in X called a *Cauchy sequence* if

$$\lim_{n\to\infty} M(X_{n+p}, x_n, t) = 1,$$

for all $t > 0$ and $p > 0$.

(c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be *complete.*

Remark 1. Since $*$ is continuous, it follows from (FM-4) that the limit of the sequence in fuzzy metric space is uniquely determined.
Let $(X, M, *)$ is a fuzzy metric space with the following condition:

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$.

Lemma 1.2 (Cho [3], Mishra, Sharma & Singh [19]). *Let $\{y_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$ with the condition* (FM-6). *If there exists a number $k \in (0, 1)$ such that*

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$$

for all $t > 0$ and $n = 1, 2, \ldots$ then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 1.3 (Mishra, Sharma & Singh [19]). *If for all $x, y \in X$, $t > 0$ and for a number $k \in (0, 1)$*

$$M(x, y, kt) \geq M(x, y, t)$$

then $x = y$.

Definition 1.4 (Mishra, Sharma & Singh [19]). Let A and B be mappings from a fuzzy metric space $(X, M, *)$ into itself. The mappings A and B are said to be *compatible* if

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1,$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$$

for some $z \in X$.

Definition 1.5 (Cho [3]). Let A and B be maps from a fuzzy metric space $(X, M, *)$ into itself. The maps A and B are said be *compatible of type* (α) if, for all $t > 0$,

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1, \text{and} \lim_{n\to\infty} M(BAx_n, AAx_n, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim Bx_n = z$$

for some $z \in X$.

Definition 1.6 (Cho, Pathak, Kang & Jung [4]). Let A and B be maps from a fuzzy metric space $(X, M, *)$ into itself. The maps A and B are said be *compatible of type* (β) if, for all $t > 0$,

$$\lim_{n\to\infty} M(AAx_n, BBx_n, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim Bx_n = z$$

for some $z \in X$.

Remark 2. In Jungck [14], Jungck, Murthy & Cho [15] and Pathak, Cho, Chang & Kang [20], we can find the equivalent formulations of Definitions 1.4, 1.5 and 1.6 and their examples in metric spaces. Such maps are independent of each other and more general than commuting and weakly commuting maps Jungck [13] and Sessa [22].

Definition 1.7 (Jungck & Rhoades [16]). Two maps A and B are said to be *weakly compatible* if they commute at a coincidence point.

Example 2. Let $X = [0, 2]$ with the metric d defined by $d(x, y) = |x - y|$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \quad x, y \in X,$$

and define

$$M(x, y, 0) = 0, \quad x, y \in X.$$

Clearly $M(x, y, *)$ is a fuzzy metric space on X where $*$ is defined by

$$a * b = ab \text{ or } a * b = \min\{a, b\}.$$

Define $A, B : X \to X$ by

$$Ax = \begin{cases} x, & \text{if } x \in [0, \frac{1}{4}), \\ \frac{1}{4}, & \text{if } x \geq \frac{1}{4}; \end{cases} \quad \text{and} \quad Bx = \frac{x}{1+x}$$

for all $x \in [0, 2]$. Consider the sequence $\{x_n = 1/3 + 1/n : n \geq 1\}$ in X. Then

$$\lim_{n\to\infty} Ax_n = \frac{1}{4}, \quad \lim_{n\to\infty} Bx_n = \frac{1}{4}.$$

But

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = \frac{t}{t + |1/4 - 1/5|} \neq 1.$$

Thus A and B are noncompatible. But A and B are commuting at their coincidence point $x = 0$, that is, weakly compatible at $x = 0$. Also

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = \frac{t}{t + |1/4 - 1/5|} \neq 1$$

and

$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) = \frac{t}{t + |1/5 - 1/4|} \neq 1.$$

Thus A and B are not compatible of type (α). Further,

$$\lim_{n\to\infty} M(AAx_n, BBx_n, t) = \frac{t}{t + |1/4 - 1/5|} \neq 1.$$

Thus A and B are not compatible of type (β). In view of this example, we observe that

(i) weakly compatible maps need not be compatible,
(ii) weakly compatible maps need not be compatible of type (α),
(iii) weakly compatible maps need not be compatible of type (β).

2. Main Results

Theorem 2.1. *Let $(X, M, *)$ be a fuzzy metric space with $t * t \geq t$ for all $t \in [0, 1]$ and the condition* (FM-6). *Let A, B, S, T, P, and Q be mappings from X into itself such that*

(a) $P(X) \subset AB(X)$, $Q(X) \subset ST(X)$,

(b) *there exists a constant* $k \in (0,1)$ *such that*

$$\begin{aligned}
&[1 + aM(STx, ABy, kt)] * M(Px, Qy, kt) \\
&\geq a[M(Px, STx, kt) * M(Qy, ABy, kt) + M(Qy, STx, kt) * M(Px, ABy, kt)] \\
&\quad + M(ABy, STx, t) * M(Px, STx, t) * M(Qy, ABy, t) * M(Qy, STx, \alpha t) \\
&\qquad * M(Px, ABy, (2-\alpha)t)
\end{aligned}$$

for all $x, y \in X$, $a \geq 0$, $\alpha \in (0,2)$ *and* $t > 0$, *and*

(c) *if one of* $P(X)$, $Q(X)$, $AB(X)$ *or* $ST(X)$ *is a complete subspace of* X,

then

(i) P *and* ST *have a coincidence point, and*

(ii) Q *and* AB *have a coincidence point.*

Further, if

(d) $AB = BA$, $QB = BQ$, $QA = AQ$, $PT = TP$ *and* $ST = TS$, *and*

(e) *the pair* $\{P, ST\}$ *is weakly compatible,*

then

(iii) A, B, S, T, P *and* Q *have a unique common fixed point in* X.

Proof. By (a), since $P(X) \subset AB(X)$, for any point $x_0 \in X$, there exists a point $x_1 \in X$ such that $Px_0 = ABx_1$. Since $Q(X) \subset ST(X)$, for this point x_1 we can choose a point $x_2 \in X$ such that $Qx_1 = STx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that, for $n = 0, 1, 2, \ldots$,

$$y_{2n} = Px_{2n} = ABx_{2n+1} \text{ and } y_{2n+1} = Qx_{2n+1} = STx_{2n+2}.$$

By (b), for all $t > 0$ and $\alpha = 1 - q$, with $q \in (0,1)$, we have

$$\begin{aligned}
&[1 + aM(y_{2n}, y_{2n+1}, kt)] * M(y_{2n+1}, y_{2n+2}, kt) \\
&= [1 + aM(STx_{2n+2}, ABx_{2n+1}, kt)] * M(Px_{2n+2}, Qx_{2n+1}, kt) \\
&\geq a[M(y_{2n+2}, y_{2n+1}, kt) * M(y_{2n+1}, y_{2n}, kt) \\
&\qquad + M(y_{2n+1}, y_{2n+1}, kt) * M(y_{2n+2}, y_{2n}, kt)] \\
&\quad + M(y_{2n}, y_{2n+1}, t) * M(y_{2n+2}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t) \\
&\qquad * M(y_{2n+1}, y_{2n+1}, (1-q)t) * M(y_{2n+2}, y_{2n}, (1+q)t) \\
&\geq a[M(y_{2n}, y_{2n+1}, kt) * M(y_{2n+1}, y_{2n+2}, kt) \\
&\qquad + M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t)] \\
&\quad * M(y_{2n}, y_{2n+1}, t) * 1 * M(y_{2n}, y_{2n+1}, qt) * M(y_{2n+1}, y_{2n+2}, t)
\end{aligned}$$

$$\geq a[M(y_{2n}, y_{2n+1}, kt) * M(y_{2n+1}, y_{2n+2}, kt) + M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t)] * M(y_{2n}, y_{2n+1}, qt).$$

Thus it follows that

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n}, y_{2n+1}, qt).$$

Since the t-norm $*$ is continuous and $M(x, y, \bullet)$ is continuous, letting $q \to 1$, we have

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t).$$

Similarly, we also have

$$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t) * M(y_{2n+2}, y_{2n+3}, t).$$

In general, we have, for $m = 1, 2, \dots,$

$$M(y_{m+1}, y_{m+2}, kt) \geq M(y_m, y_{m+1}, t) * M(y_{m+1}, y_{m+2}, t).$$

Consequently, it follows that, for $m = 1, 2, \dots; p = 1, 2, \dots,$

$$M(y_{m+1}, y_{m+2}, kt) \geq M(y_m, y_{m+1}, t) * M\Big(y_{m+1}, y_{m+2}, \frac{t}{k^p}\Big).$$

By nothing that $*M(y_{m+1}, y_{m+2}, \frac{t}{k^p}) \to 1$ as $p \to \infty$, we have, for $m = 1, 2, \dots,$

$$M(y_{m+1}, y_{m+2}, kt) \geq M(y_m, y_{m+1}, t).$$

Hence, by Lemma 1.2, $\{y_n\}$ is a Cauchy sequence in X.

Now suppose $ST(X)$ is complete. Note that the subsequence $\{y_{2n+1}\}$ is contained in $ST(X)$ and has a limit in $ST(X)$. Call it z.

Let $u \in ST^{-1}z$. Then $STu = z$. We shall use the fact that the subsequence $\{y_{2n}\}$ also converges to z. By (b), with $\alpha = 1$, we have

$$[1 + aM(STu, y_{2n}, kt)] * M(Pu, y_{2n+1}, kt)$$
$$\geq a[M(Pu, STu, kt) * M(y_{2n+1}, y_{2n}, kt) + M(y_{2n+1}, STu, kt) * M(Pu, y_{2n}, kt)] + M(y_{2n}, STu, t) * M(Pu, STu, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n+1}, STu, t) * M(Pu, y_{2n}, t)$$

which implies that, as $n \to \infty$,

$$M(Pu, z, kt) \geq M(Pu, z, t).$$

Therefore, by Lemma 1.3, we have $Pu = z$. Since $STu = z$ thus $Pu = z = STu$, *i. e.*, u is a coincidence point of P and ST. This proves (i).

Since $P(X) \subset AB(X)$, $Pu = z$ implies that $z \in AB(X)$. Let $v \in AB^{-1}z$. Then $ABv = z$.

By (b), with $\alpha = 1$, we have

$$[1 + aM(y_{2n+1}, ABv, kt)] * M(y_{2n}, Qv, kt)$$
$$\geq a[M(y_{2n+2}, y_{2n+1}, kt) * M(Qv, ABv, kt) + M(Qv, y_{2n+1}, kt) * M(y_{2n+2}, ABv, kt)]$$
$$+ M(ABv, y_{2n+1}, t) * M(y_{2n+2}, y_{2n+1}, t) * M(Qv, ABv, t) * M(Qv, y_{2n+1}, t)$$
$$* M(y_{2n+2}, ABv, t),$$

which implies that, as $n \to \infty$,

$$M(z, Qv, kt) \geq M(z, Qv, t).$$

Therefore, by Lemma 1.3, we have $Qv = z$. Since $ABv = z$, we have $Qv = z = ABv$, *i. e.*, v is coincidence point of Q and AB. This proves (ii).

The remaining two cases pertain essentially to the previous cases. Indeed if $P(X)$ or $Q(X)$ is complete, then by (a) $z \in P(X) \subset AB(X)$ or $z \in Q(X) \subset ST(X)$. Thus (i) and (ii) are completely established.

Since the pair $\{P, ST\}$ is weakly compatible therefore P and ST commute at their coincidence point, *i. e.*, $P(STu) = (ST)Pu$ or $Pz = STz$. By (d), we have

$$Q(ABv) = (AB)Qv \text{ or } Qz = ABz.$$

Now we, prove that $Pz = z$, By (b), with $\alpha = 1$, we have

$$[1 + aM(STz, y_{2n}, kt)] * M(Pz, y_{2n+1}, kt)$$
$$\geq a[M(Pz, STz, kt) * M(y_{2n+1}, y_{2n}, kt) + M(y_{2n+1}, STz, kt) * M(Pz, y_{2n}, kt)]$$
$$+ M(y_{2n}, STz, t) * M(Pz, STz, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n+1}, STz, t)$$
$$* M(Pz, y_{2n}, t)$$

Proceeding limit as $n \to \infty$, we have

$$M(Pz, z, kt) \geq M(Pz, z, t).$$

Therefore, by Lemma 1.3, we have $Pz = z$ so $Pz = STz = z$. By (b), with $\alpha = 1$, we have

$$[1 + aM(y_{2n+1}, ABz, kt)] * M(y_{2n+2}, Qz, kt)$$
$$\geq a[M(y_{2n+2}, y_{2n+1}, kt) * M(Qz, ABz, kt) + M(Qz, y_{2n+1}, kt) * M(y_{2n+2}, ABz, kt)]$$
$$+ M(ABz, y_{2n+1}, t) * M(y_{2n+2}, y_{2n+1}, t) * M(Qz, ABz, t) * M(Qz, y_{2n+1}, t)$$
$$* M(y_{2n+2}, ABz, t).$$

Proceeding limit as $n \to \infty$, we have

$$M(z, Qz, kt) \geq M(Qz, z, t).$$

Therefore, by Lemma 1.3, we have $Qz = z$ so $Qz = ABz = z$. By (b), with $\alpha = 1$, and using (d), we have

$$\begin{aligned}
&[1 + aM(STz, AB(Bz), kt)] * M(Pz, Q(Bz), kt) \\
&\quad \geq a[M(Pz, STz, kt) * M(Q(Bz), AB(Bz), kt) \\
&\qquad\qquad + M(Q(Bz), STz, kt) * M(Pz, AB(Bz), kt)] \\
&\quad + M(AB(Bz), STz, t) * M(Pz, STz, t) * M(Q(Bz), AB(Bz), t) \\
&\qquad\qquad * M(Q(Bz), STz, t) * M(Pz, AB(Bz), t).
\end{aligned}$$

Thus, we have

$$M(z, Bz, kt) \geq M(Bz, z, t) * 1 * 1 * M(Bz, z, t) * M(z, Bz, t) \geq M(Bz, z, t).$$

Therefore, by Lemma 1.3, we have $Bz = z$. Since $ABz = z$, therefore $Az = z$.

Again by (b), with $\alpha = 1$, and using (d), we have

$$\begin{aligned}
&[1 + aM(ST(Tz), ABz, kt)] * M(P(Tz), Qz, kt) \\
&\quad \geq a[M(P(Tz), ST(Tz), kt) * M(Qz, ABz, kt) + M(Qz, ST(Tz), kt) \\
&\qquad\qquad * M(P(Tz), ABz, kt)] \\
&\quad + M(ABz, ST(Tz), t) * M(P(Tz), ST(Tz), t) * M(Qz, ABz, t) \\
&\qquad\qquad * M(Qz, ST(Tz), t) * M(P(Tz), ABz, t).
\end{aligned}$$

Thus, it follows that

$$M(Tz, z, kt) \geq M(z, Tz, t) * 1 * 1 * M(z, Tz, t) * M(z, Tz, t).$$

Therefore, by Lemma 1.3, we have $Tz = z$. Since $STz = z$, therefore $Sz = z$.

By combining the above results, we have

$$Az = Bz = Sz = Tz = Qz = z,$$

that is, z is a common fixed point of A, B, S, T, P, and Q. The uniqueness of the common fixed point of A, B, S, T, P, and Q follows easily from (b). This completes the proof. □

From Theorem 2.1, with $a = 0$, we have the following result.

Corollary 2.2. *Let $(X, M, *)$ be a fuzzy metric space with $t * t \geq t$ for all $t \in [0,1]$ and the condition* (FM-6). *Let A, B, S, T, P, and Q be mappings from X into itself such that*

(a) $P(X) \subset AB(X), Q(X) \subset ST(X)$,

(b) *there exists a constant $k \in (0,1)$ such that*

$$M(Px, Qy, kt) \geq M(ABy, STx, t) * M(Px, STx, t) * M(Qy, ABy, t) * M(Qy, STx, t) * M(Px, ABy, (2-\alpha)t)$$

for all $x, y \in X$, $\alpha \in (0,2)$ and $t > 0$, and

(c) *if one of $P(X)$, $Q(X)$, $AB(X)$ or $ST(X)$ is a complete subspace of X,*

then

(i) *P and ST have a coincidence point, and*

(ii) *Q and AB have a coincidence point.*

Further, if

(d) *$AB = BA$, $QB = BQ$, $QA = AQ$, $PT = TP$ and $ST = TS$, and*

(e) *the pair $\{P, ST\}$ is weakly compatible,*

then

(iii) *A, B, S, T, P, and Q have a unique common fixed point in X.*

Remark 3. Theorem 2.1 and Corollary 2.2 improve and extend results of Mishra, Sharma & Singh [19], Cho [3] and Sharma & Deshpande [26]. Theorem 2.1 and Corollary 2.2 also improve the results of Cho, Pathak, Kang & Jung [4], Sharma [24] and Sharma & Deshpande [25].

If we put $P = Q$ in Theorem 2.1, we have the following result.

Corollary 2.3. *Let $(X, M, *)$ be a fuzzy metric space with $t * t \geq t$ for all $t \in [0,1]$ and the condition* (FM-6). *Let A, B, S, T, and P be mappings from X into itself such that*

(a) $P(X) \subset AB(X), P(X) \subset ST(X)$,

(b) *there exists a constant $k \in (0,1)$ such that*

$$[1 + aM(STx, ABy, kt)] * M(Px, Py, kt) \geq a[M(Px, STx, kt) * M(Py, ABy, kt) + M(Py, STx, kt) * M(Px, ABy, kt)]$$

$$+ M(ABy, STx, t) * M(Px, STx, t) * M(Py, ABy, t) * M(Py, STx, \alpha t) * M(Px, ABy, (2-\alpha)t)$$

for all $x, y \in X$, $a \geq 0$, $\alpha \in (0,2)$ *and* $t > 0$, *and*

(c) *if one of* $P(X)$, $AB(X)$ *or* $ST(X)$ *is a complete subspace of* X,

then

(i) P *and* ST *have a coincidence point, and*

(ii) P *and* AB *have a coincidence point.*

Further, if

(d) $AB = BA, PB = BP, PA = AP, PT = TP$ *and* $ST = TS$, *and*

(e) *the pair* $\{P, ST\}$ *is weakly compatible,*

then

(iii) A, B, S, T, *and* P *have a unique common fixed point in* X.

From Corollary 2.3, with $a = 0$, we have the following:

Corollary 2.4. *Let* $(X, M, *)$ *be a fuzzy metric space with* $t * t \geq t$ *for all* $t \in [0,1]$ *and the condition* (FM-6). *Let* A, B, S, T, *and* P *be mappings from* X *into itself such that*

(a) $P(X) \subset AB(X), P(X) \subset ST(X)$,

(b) *there exists a constant* $k \in (0,1)$ *such that*

$$M(Px, Py, kt) \geq M(ABy, STx, t) * M(Px, STx, t) * M(Py, ABy, t) * M(Py, STx, \alpha t) * M(Px, ABy, (2-\alpha)t)$$

for all $x, y \in X$, $\alpha \in (0,2)$ *and* $t > 0$, *and*

(c) *if one of* $P(X)$, $AB(X)$ *or* $ST(X)$ *is a complete subspace of* X,

then

(i) P *and* ST *have a coincidence point, and*

(ii) P *and* AB *have a coincidence point.*

Further, if

(d) $AB = BA, PB = BP, PA = AP, PT = TP$ *and* $ST = TS$, *and*

(e) *the pair* $\{P, ST\}$ *is weakly compatible,*

then

(iii) *A, B, S, T and P have a unique common fixed point in X.*

Remark 4. Corollaries 2.3 and 2.4 improve and extend the results of Mishra, Sharma & Singh [19] and Sharma & Deshpande [26]. Corollaries 2.3 and 2.4 also improve the result of Cho [3].

If we put $B = T = I_X$ (the identity mapping on X) in Theorem 2.1, we have the following result.

Corollary 2.5. *Let $(X, M, *)$ be a fuzzy metric space with $t * t \geq t$ for all $t \in [0, 1]$ and the condition* (FM-6). *Let A, S, P and Q be mappings from X into itself such that*

(a) $P(X) \subset A(X), Q(X) \subset S(X)$,

(b) *there exists a constant $k \in (0, 1)$ such that*

$$\begin{aligned}
&[1 + aM(Sx, Ay, kt)] * M(Px, Qy, kt) \\
&\quad \geq a[M(Px, Sx, kt) * M(Qy, Ay, kt) + M(Qy, Sx, kt) * M(Px, Ay, kt)] \\
&\qquad + M(Ay, Sx, t) * M(Px, Sx, t) * M(Qy, Ay, t) * M(Qy, Sx, \alpha t) \\
&\qquad\qquad * M(Px, Ay, (2 - \alpha)t)
\end{aligned}$$

for all $x, y \in X$, $a \geq 0$, $\alpha \in (0, 2)$ and $t > 0$, and

(c) *if one of $P(X), Q(X), A(X)$ or $S(X)$ is a complete subspace of X,*

then

(i) *P and S have a coincidence point, and*

(ii) *Q and A have a coincidence point.*

Further, if

(d) *$QA = AQ$, and*

(e) *the pair $\{P, S\}$ is weakly compatible,*

then

(iii) *A, S, P and Q have a unique common fixed point in X.*

From Corollary 2.5, with $a = 0$, we have the following result.

Corollary 2.6. *Let $(X, M, *)$ be a fuzzy metric space with $t * t \geq t$ for all $t \in [0, 1]$ and the condition* (FM-6). *Let A, S, P and Q be mappings from X into itself such that*

(a) $P(X) \subset A(X), Q(X) \subset S(X)$,

(b) *there exists a constant* $k \in (0,1)$ *such that*

$$\begin{aligned} &M(Px, Qy, kt) \\ &\quad \geq M(Ay, Sx, t) * M(Px, Sx, t) * M(Qy, Ay, t) * M(Qy, Sx, \alpha t) \\ &\qquad * M(Px, Ay, (2-\alpha)t) \end{aligned}$$

for all $x, y \in X$, $\alpha \in (0,2)$ *and* $t > 0$, *and*

(c) *if one of P(X), Q(X), A(X) or S(X) is a complete subspace of X,*

then

(i) *P and S have a coincidence point, and*

(ii) *Q and A have a coincidence point.*

Further, if

(d) $QA = AQ$, *and*

(e) *the pair* $\{P, S\}$ *is weakly compatible,*

then

(iii) *A, S, P and Q have a unique common fixed point in X.*

Remark 5. Corollaries 2.5 and 2.6 improve the results of Mishra, Sharma & Singh [19] and Sharma & Deshpande [26].

If we put $A = S$ in Corollary 2.5, we have the following result.

Corollary 2.7. *Let* $(X, M, *)$ *be a fuzzy metric space with* $t * t \geq t$ *for all* $t \in [0,1]$ *and the condition* (FM-6). *Let A, P and Q be mappings from X into itself such that*

(a) $P(X) \subset A(X), Q(X) \subset A(X)$,

(b) *there exists a constant* $k \in (0,1)$ *such that*

$$\begin{aligned} &[1 + aM(Ax, Ay, kt)] * M(Px, Qy, kt) \\ &\quad \geq a[M(Px, Ax, kt) * M(Qy, Ay, kt) + M(Qy, Ax, kt) * M(Px, Ay, kt)] \\ &\qquad + M(Ay, Ax, t) * M(Px, Ax, t) * M(Qy, Ay, t) * M(Qy, Ax, \alpha t) \\ &\qquad\qquad * M(Px, Ay, (2-\alpha)t) \end{aligned}$$

for all $x, y \in X$, $a \geq 0$, $\alpha \in (0,2)$ *and* $t > 0$, *and*

(c) *if one of* $P(X)$, $Q(X)$, $A(X)$ *is a complete subspace of* X,

then

(i) *P and S have a coincidence point, and*

(ii) *Q and A have a coincidence point.*

Further, if

(d) $QA = AQ$, *and*

(e) *the pair* $\{P, A\}$ *is weakly compatible,*

then

(iii) A, P *and* Q *have a unique common fixed point in* X.

From Corollary 2.7, with $a = 0$, we have the following result.

Corollary 2.8. *Let* $(X, M, *)$ *be a fuzzy metric space with* $t * t \geq t$ *for all* $t \in [0, 1]$ *and the condition* (FM-6). *Let* A, P *and* Q *be mappings from* X *into itself such that*

(a) $P(X) \subset A(X), Q(X) \subset A(X)$,

(b) *there exists a constant* $k \in (0, 1)$ *such that*

$$M(Px, Qy, kt) \geq M(Ay, Ax, t) * M(Px, Ax, t) * M(Qy, Ay, t) * M(Qy, Ax, \alpha t) * M(Px, Ay, (2 - \alpha)t)$$

for all $x, y \in X$, $\alpha \in (0, 2)$ *and* $t > 0$, *and*

(c) *if one of* $P(X)$, $Q(X)$, $A(X)$ *is a complete subspace of* X,

then

(i) P *and* A *have a coincidence point, and*

(ii) Q *and* A *have a coincidence point.*

Further, if

(d) $QA = AQ$, *and*

(e) *the pair* $\{P, A\}$ *is weakly compatible,*

then

(iii) A, P *and* Q *have a unique common fixed point in* X.

In Theorem 2.1, if we replace the condition $QA = AQ$ by weak compatibility of the pair $\{Q, AB\}$ then we have the following theorem.

Theorem 2.9. *Let* $(X, M, *)$ *be a fuzzy metric space with* $t * t \geq t$ *for all* $t \in [0, 1]$ *and the condition* (FM-6). *Let* A, B, S, T, P *and* Q *be mappings from* X *into itself such that*

(a) $P(X) \subset AB(X), Q(X) \subset ST(X)$,

(b) *there exists a constant* $k \in (0,1)$ *such that*

$$
\begin{aligned}
&[1 + aM(STx, ABy, kt)] * M(Px, Qy, kt) \\
&\geq a[M(Px, STx, kt) * M(Qy, ABy, kt) + M(Qy, STx, kt) * M(Px, ABy, kt)] \\
&\quad + M(ABy, STx, t) * M(Px, STx, t) * M(Qy, ABy, t) * M(Qy, STx, \alpha t) \\
&\qquad\qquad * M(Px, ABy, (2-\alpha)t)
\end{aligned}
$$

for all $x, y \in X, a \geq 0, \alpha \in (0,2)$ *and* $t > 0$, *and*

(c) *if one of* $P(X)$, $Q(X)$, $AB(X)$ *or* $ST(X)$ *is a complete subspace of* X,

then

(i) P *and* ST *have a coincidence point, and*

(ii) Q *and* AB *have a coincidence point.*

Further, if

(d) $AB = BA, QB = BQ, PT = TP$ *and* $ST = TS$, *and*

(e) *the pairs* $\{P, ST\}$ *and* $\{Q, AB\}$ *are weakly compatible,*

then

(iii) A, B, S, T, P *and* Q *have a unique common fixed point in* X.

By using Theorem 2.9, we have the following theorem.

Theorem 2.10. *Let* $(X, M, *)$ *be a fuzzy metric space with* $t * t \geq t$ *for all* $t \in [0,1]$ *and the condition* (FM-6). *Let* A, B, S, T *and* P_i, *for* $i = 0, 1, 2, \ldots$, *be mappings from* X *into itself such that*

(a) $P_0(X) \subset AB(X), P_i(X) \subset ST(X)$, *for* $i \in \mathbb{N}$,

(b) *there exists a constant* $k \in (0,1)$ *such that*

$$
\begin{aligned}
&[1 + aM(STx, ABy, kt)] * M(P_0x, Qy, kt) \\
&\geq a[M(P_0x, STx, kt) * M(P_iy, ABy, kt) + M(P_iy, STx, kt) * M(P_0x, ABy, kt)] \\
&\quad + M(ABy, STx, t) * M(P_0x, STx, t) * M(P_iy, ABy, t) * M(P_iy, STx, \alpha t) \\
&\qquad\qquad * M(P_0x, ABy, (2-\alpha)t)
\end{aligned}
$$

for all $x, y \in X$, $a \geq 0$, $\alpha \in (0,2)$ *and* $t > 0$, *and*

(c) *if one of* $P_0(X), AB(X)$ *or* $ST(X)$ *is a complete subspace of* X *or alternatively,* P_i, *for* $i \in \mathbb{N}$, *are complete subspaces of* X,

then

(i) P_0 *and* ST *have a coincidence point, and*

(ii) *for* $i \in \mathbb{N}$, P_i *and* AB *have a coincidence point.*

Further, if

(d) $AB = BA, P_iB = BP_i (i \in \mathbb{N}), P_0T = TP_0$ *and* $ST = TS$, *and*

(e) *the pairs* $\{P_0, ST\}$ *and* $P_i, AB (i \in \mathbb{N})$ *are weakly compatible,*

then

(iii) A, B, S, T *and* P_i, *for* $i = 0, 1, 2, \ldots$, *have a unique common fixed point in* X.

From Theorem 2.9, with $a = 0$, we have the following result due to Sharma & Deshpande [27].

Corollary 2.11. *Let* $(X, M, *)$ *be a fuzzy metric space with* $t * t \geq t$ *for all* $t \in [0, 1]$ *and the condition* (FM-6). *Let* A, B, S, T, P *and* Q *be mappings from* X *into itself such that*

(a) $P(X) \subset AB(X), Q(X) \subset ST(X)$,

(b) *there exists a constant* $k \in (0, 1)$ *such that*

$$M(Px, Qy, kt) \geq M(ABy, STx, t) * M(Px, STx, t) * M(Qy, ABy, t) * M(Qy, STx, \alpha t) * M(Px, ABy, (2 - \alpha)t)$$

for all $x, y \in X, \alpha \in (0, 2)$ *and* $t > 0$, *and*

(c) *if one of* $P(X), Q(X), AB(X)$ *or* $ST(X)$ *is a complete subspace of* X,

then

(i) P *and* ST *have a coincidence point, and*

(ii) Q *and* AB *have a coincidence point.*

Further, if

(d) $AB = BA, QB = BQ, PT = TP$ *and* $ST = TS$, *and*

(e) *the pairs* $\{P, ST\}$ *and* $\{Q, AB\}$ *are weakly compatible,*

then

(iii) A, B, S, T, P *and* Q *have a unique common fixed point in* X.

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