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SOME NEW MEASURES OF FUZZY DIRECTED DIVERGENCE AND THEIR GENERALIZATION

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ABSTRACT. There exist many measures of fuzzy directed divergence corresponding to the existing probabilistic measures. Some new measures of fuzzy divergence have been proposed which correspond to some well-known existing probabilistic measures. The essential properties of the proposed measures have been developed which contains many existing measures of fuzzy directed divergence.

1. INTRODUCTION

Zadeh [8, 9] introduced the concept of fuzzy sets in which imprecise knowledge can be used to define a event. Dubois & Prade [2] defined the distance between two fuzzy subsets on a fuzzy subset of $\mathbb{R}^+$. Their definition does not generalize the shortest distance between two crisp sets. Rosenfeld [6] defined the shortest distance between two fuzzy sets as a density function on the non-negative reals, which generalizes the definition of shortest distance for crisp sets in a natural way. Using the concept of fuzzy message conditioning, a fuzzy information measure for discrimination between two fuzzy sets has been suggested by Bhandari & Pal [1].

Bhandari & Pal [1] introduced a measure of fuzzy divergence corresponding to the probabilistic directed divergence of Kullback & Leibler [4] which is given by

$$I(A:B)=\sum_{i=1}^{n}\left[\mu_A(x_i)\ln\frac{\mu_A(x_i)}{\mu_B(x_i)}+(1-\mu_A(x_i))\ln\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}\right], \tag{1.1}$$

where $\mu_A(x_i)$ gives the degree of belongingness of the element x_i to the set A.

The symmetric fuzzy divergence between two fuzzy sets A and B is given by

$$J(A:B)=I(A:B)+I(B:A). \tag{1.2}$$

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Lin [5] introduced a probabilistic measure

$$K(P:Q)=\sum_{i=1}^{n}p_i\ln\frac{p_i}{(p_i+q_i)/2} \tag{1.3}$$

of directed divergence of a probability distribution $P=(p_1,p_1,\dots,p_n)$ from another probability distribution $Q=(q_1,q_2,\dots,q_m)$.

We propose fuzzy directed divergence corresponding to (1.3) as

$$\begin{aligned}K(A:B)&\\=\sum_{i=1}^{n}\Bigg[\mu_A(x_i)\ln&\frac{\mu_A(x_i)}{(\mu_A(x_i)+\mu_B(x_i))/2}\\&+(1-\mu_A(x_i))\ln\frac{1-\mu_A(x_i)}{(1-\mu_A(x_i)+1-\mu_B(x_i))/2}\Bigg].\end{aligned} \tag{1.4}$$

The probabilistic measure of directed divergence of Sharma & Taneja [7] is given by

$$S(P:Q)=\frac{1}{\alpha-\beta}\sum_{i=1}^{n}\left[p_i^{\alpha}\,q_i^{1-\alpha}-p_i^{\beta}\,q_i^{1-\beta}\right] \tag{1.5}$$

where $\alpha<1$, $\beta>1$ or $\alpha>1$, $\beta<1$.

We propose the measure of fuzzy directed divergence corresponding to (1.5) as

$$\begin{aligned}S(A:B)&\\=\frac{1}{\alpha-\beta}\sum_{i=1}^{n}\Big[&\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i)^{1-\alpha}\\&-\mu_A^{\beta}(x_i)\mu_B^{1-\beta}(x_i)-(1-\mu_A(x_i))^{\beta}(1-\mu_B(x_i))^{1-\beta}\Big]\end{aligned} \tag{1.6}$$

where $\alpha<1$, $\beta>1$ or $\alpha>1$, $\beta<1$.

In Section 2, we prove that (1.4) and (1.6) are valid measures of fuzzy directed divergence. In Section 3, a new generalized measure of fuzzy directed divergence has been developed and many more well-known measures have been derived from it.

2. NEW MEASURES OF FUZZY DIVERGENCE

Taking

$$\sum_{i=1}^{n}\mu_A(x_i)=s \quad\text{and}\quad \sum_{i=1}^{n}\mu_B(x_i)=t,$$

where s and t may be different from unity.

We know that

$$\sum_{i=1}^{n} \frac{\mu_A(x_i)}{s} \ln \frac{\frac{\mu_A(x_i)}{s}}{\frac{\mu_A(x_i)+\mu_B(x_i)}{2} / \frac{s+t}{2}} \geq 0$$

that is,

$$\sum_{i=1}^{n} \mu_A(x_i) \ln \frac{\mu_A(x_i)}{(\mu_A(x_i) + \mu_B(x_i))/2} \geq s \ln \frac{s}{(s+t)/2}. \tag{2.1}$$

Similarly,

$$\sum_{i=1}^{n} (1 - \mu_A(x_i)) \ln \frac{1 - \mu_A(x_i)}{(1 - \mu_A(x_i) + 1 - \mu_B(x_i))/2} \geq (n-s) \ln \frac{n-s}{(2n - s + t)/2}. \tag{2.2}$$

Adding (2.1) and (2.2), we get

$$K(A : B) \geq f(s),$$

where

$$f(s) = s \ln \frac{s}{(s+t)/2} + (n-s) \ln \frac{n-s}{(2n-s-t)/2}.$$

Now

$$f'(s) = \ln s - \frac{s}{s+t} - \ln(s+t) - \ln(n-s) + \frac{n-s}{2n-s-t} + \ln(2n-s-t)$$

and

$$\begin{aligned} f''(s) &= \left(\frac{1}{s} - \frac{1}{s+t}\right) - \frac{t}{(s+t)^2} + \left(\frac{1}{n-s} - \frac{1}{2n-s-t}\right) - \frac{n-t}{(2n-s-t)^2} \\ &= \frac{t^2}{s(s+t)^2} + \frac{(n-t)^2}{(n-s)(2n-s-t)^2} \\ &> 0, \end{aligned}$$

so that $f(s)$ is a convex function of s which has its minimum value when $s = t$, and the minimum value is 0 so that $f(s) > 0$ and vanishes when $s = t$. Consequently, $K(A : B)$ is a convex function of $\mu_A(x_i)$.

Similarly, we can show that $K(A : B)$ is a convex function of $\mu_B(x_i)$. Thus for all values of s and t, we have

(i) $K(A : B) \geq 0$,
(ii) $K(A : B) = 0$ if and only if $A = B$,
(iii) $K(A : B)$ is a convex function, and
(iv) $K(A : B)$ does not change when $\mu_A(x_i)$ is changed to $1 - \mu_A(x_i)$ and $\mu_B(x_i)$ is changed to $1 - \mu_B(x_i)$.

Hence $K(A:B)$ is a valid measure of fuzzy directed divergence and

$$J'(A:B) = K(A:B) + K(B:A)$$

is a valid measure of fuzzy symmetric divergence.

Again, we know that

$$\sum_{i=1}^{n}\left[\left(\frac{\mu_A(x_i)}{s}\right)^{\alpha}\left(\frac{\mu_B(x_i)}{t}\right)^{1-\alpha} - 1\right] \geq 0,$$

that is,

$$\sum_{i=1}^{n}\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i) \geq s^{\alpha}t^{1-\alpha}. \tag{2.3}$$

Similarly,

$$\sum_{i=1}^{n}(1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha} \geq (n-s)^{\alpha}(n-t)^{1-\alpha}. \tag{2.4}$$

Adding (2.3) and (2.4), we get

$$\sum_{i=1}^{n}\left[\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha}\right] \geq s^{\alpha}t^{1-\alpha} + (n-s)^{\alpha}(n-t)^{1-\alpha}. \tag{2.5}$$

Similarly,

$$\sum_{i=1}^{n}\left[\mu_A^{\beta}(x_i)\mu_B^{1-\beta}(x_i) + (1-\mu_A(x_i))^{\beta}(1-\mu_B(x_i))^{1-\beta}\right] \geq s^{\beta}t^{1-\beta} + (n-s)^{\beta}(n-t)^{1-\beta}. \tag{2.6}$$

Subtracting (2.6) from (2.5) and dividing by $(\alpha - \beta)$, we get

$$S(A:B) \geq F(s)$$

where

$$F(s) = \frac{1}{\alpha-\beta}\left[s^{\alpha}t^{1-\alpha} + (n-s)^{\alpha}(n-t)^{1-\alpha} - s^{\beta}t^{1-\beta} - (n-s)^{\beta}(n-t)^{1-\beta}\right].$$

Now we have

$$F'(s) = \frac{1}{\alpha-\beta}\left[\alpha\left(\frac{s}{t}\right)^{\alpha-1} - \alpha\left(\frac{n-s}{n-t}\right)^{\alpha-1}\right]$$

and

$$F''(s) = \frac{1}{\alpha-\beta}\left[\frac{\alpha(\alpha-1)}{t}\left(\frac{s}{t}\right)^{\alpha-2} + \frac{\alpha(\alpha-1)}{n-t}\left(\frac{n-s}{n-t}\right)^{\alpha-2}\right]$$

$$= \frac{\alpha(\alpha-1)}{\alpha-\beta}\left[\frac{1}{t}\left(\frac{s}{t}\right)^{\alpha-2}+\frac{1}{n-t}\left(\frac{n-s}{n-t}\right)^{\alpha-2}\right]$$
$$> 0,$$

for $\alpha > 1$, $\beta < 1$ or $\alpha < 1$, $\beta > 1$. Hence $F(s)$ is a convex function of s whose minimum value arises when

$$\frac{s}{t} = \frac{n-s}{n-t} = 1, \quad i.\,e., \quad s = t$$

and the minimum value is 0, so that $F(s) > 0$ and vanishes only when $s = t$, *i. e.*, when $A = B$. Consequently, $S(A:B)$ is a convex function of $\mu_A(x_i)$.

Similarly, $S(A:B)$ is a convex function of $\mu_B(x_i)$. Thus, for all values of s and t, we have

(i) $S(A:B) \geq 0$,
(ii) $S(A:B) = 0$ if and only if $A = B$,
(iii) $S(A:B)$ is convex function, and
(iv) $S(A:B)$ does not change when $\mu_A(x_i)$ is replaced by $1-\mu_A(x_i)$ and $\mu_B(x_i)$ by $1-\mu_B(x_i)$.

Hence $S(A:B)$ is a valid of measure of fuzzy directed divergence and $S'(A:B) = S(A:B)+S(B:A)$ is a valid measure of fuzzy symmetric divergence corresponding to the measure defined by Havrda & Charvát [3].

If we take $\beta = 1$ and $\alpha \to 1$, $S(A:B)$ becomes the measure of fuzzy directed divergence defined by Bhandari & Pal [1].

3. Generalized Fuzzy Directed Divergence

We consider

$$\begin{aligned} I_\lambda(A:B) \\ = \sum_{i=1}^{n}\Bigg[&(\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i))\phi\Big(\frac{\mu_A(x_i)}{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)}\Big) \\ &+\big\{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))\big\}\times \\ &\phi\left(\frac{1-\mu_A(x_i)}{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))}\right)\Bigg], \end{aligned} \tag{3.1}$$

where $\phi(\cdot)$ is twice differentiable convex function for which $\phi(1) = 0$. Now

$$\frac{\partial I_\lambda(A:B)}{\partial \mu_A(x_i)}$$

$$= \lambda\phi\left(\frac{\mu_A(x_i)}{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)}\right) + \frac{(1-\lambda)\mu_B(x_i)}{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)}\phi'\left(\frac{\mu_A(x_i)}{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)}\right) - \lambda\phi\left(\frac{1-\mu_A(x_i)}{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))}\right) - \frac{(1-\lambda)(1-\mu_B(x_i))}{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-v\mu_B(x_i))}\phi\left(\frac{1-\mu_A(x_i)}{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))}\right)$$

and

$$\frac{\partial^2 I_\lambda(A:B)}{\partial\mu_A^2(x_i)} = \frac{\lambda^2(1-\lambda)^2(1-\mu_B(x_i))^2}{[\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))]^3} + \frac{(1-\lambda)^2\mu_B^2(x_i)}{[\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)]^3} > 0,$$

so that $I_\lambda(A:B)$ is a convex function of $\mu_A(x_i)$ which has its minimum value when $\mu_A(x_i) = \mu_B(x_i)$. And the minimum value is 0 so that $I_\lambda(A:B) > 0$ and vanishes when $\mu_A(x_i) = \mu_B(x_i)$. Similarly, $I_\lambda(A:B)$ is a convex function of $\mu_B(x_i)$.

Thus for all values of $\mu_A(x_i)$ and $\mu_B(x_i)$, we have

(i) $I_\lambda(A:B) \geq 0$,
(ii) $I_\lambda(A:B) = 0$ if and only if $A = B$,
(iii) $I_\lambda(A:B)$ is a convex function, and
(iv) $I_\lambda(A:B)$ does not change when $\mu_A(x_i)$ is replaced by $1-\mu_A(x_i)$ and $\mu_B(x_i)$ by $1-\mu_B(x_i)$.

Hence $I_\lambda(A:B)$ is a valid generalized measure of fuzzy directed divergence.

3.1. Special Case I: Taking $\phi(x) = x\ln x$ and denoting $I_\lambda(A:B)$ in (3.1) by $I_{1,\lambda}(A:B)$, we get

$$I_{1,\lambda}(A:B) = \sum_{i=1}^{n}\left[\mu_A(x_i)\ln\frac{\mu_A(x_i)}{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)} + (1-\mu_A(x_i))\ln\frac{1-\mu_A(x_i)}{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))}\right] \tag{3.2}$$

The expression (3.2) is a generalization of (1.4).

(a) If we take $\lambda = 0$ in (3.2), we get

$$I_{1,0}(A:B) = \sum_{i=1}^{n}\left[\mu_A(x_i)\ln\frac{\mu_A(x_i)}{\mu_B(x_i)} + (1-\mu_A(x_i))\ln\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}\right] \tag{3.3}$$

which is a measure of fuzzy directed divergence corresponding to the probabilistic measure of divergence introduced by Kullback & Leibler [4].

(b) If we take $\lambda = \frac{1}{2}$ in (3.2), we get

$$I_{1,\frac{1}{2}}(A:B) = \sum_{i=1}^{n}\left[\mu_A(x_i)\ln\frac{\mu_A(x_i)}{(\lambda\mu_A(x_i)+\mu_B(x_i))/2} + (1-\mu_A(x_i))\ln\frac{1-\mu_A(x_i)}{(1-\mu_A(x_i)+1-\mu_B(x_i))/2}\right]$$

which is (1.4).

3.2. Special Case II: Let $\phi(x) = \frac{x^\alpha - x}{\alpha(\alpha-1)}$, $\alpha \neq 0$, $\alpha \neq 1$, and denote $I_\lambda(A:B)$ in (3.1) by $I_{2,\lambda}(A:B)$, then (3.1) gives

$$I_{2,\lambda}(A:B) = \frac{1}{\alpha(\alpha-1)}\sum_{i=1}^{n}\Big[\mu_A^\alpha(x_i)\{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)\}^{1-\alpha} + (1-\mu_A(x_i))^\alpha\{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))\}^{1-\alpha} - 1\Big]. \tag{3.4}$$

(a) If we take $\lambda = 0$ in (3.4), we get

$$I_{2,0}(A:B) = \frac{1}{\alpha(\alpha-1)}\sum_{i=1}^{n}\left[\mu_A^\alpha(x_i)\mu_B(x_i)^{1-\alpha} + (1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha} - 1\right]$$

which is a measure of fuzzy directed divergence corresponding to the probabilistic divergence defined by Havrda & Charvát [3].

(b) From (3.4), we have

$$\lim_{\alpha\to 1} I_{2,\lambda}(A:B) = I_{1,\lambda}(A:B).$$

(c) When $\lambda = \frac{1}{2}$ and $\alpha \to 1$, (3.4) becomes $K(A:B)$.

(d) If we take $\lambda = 0$, $\alpha \to 1$ in (3.4), we get the measure of fuzzy directed divergence defined by Bhandari & Pal [1].

3.3. Special Case III: If we take $\phi(x) = \frac{x^\alpha - x^\beta}{\alpha-\beta}$ and denote $I_\lambda(A:B)$ in (3.1) by $I_{3,\lambda}(A:B)$, we get

$$\begin{aligned} I_{3,\lambda}&(A:B)\\ &= \frac{1}{\alpha-\beta}\sum_{i=1}^{n}\Big[\mu_A^\alpha(x_i)\{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)\}^{1-\alpha}\\ &\qquad + (1-\mu_A(x_i))^\alpha\{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))\}^{1-\alpha}\\ &\qquad - \mu_A^\beta(x_i)\{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)\}^{1-\beta}\\ &\qquad - (1-\mu_A(x_i))^\beta\{\lambda(1-\mu_A(x_i))+(1-\lambda)(1-\mu_B(x_i))\}^{1-\beta}\Big] \end{aligned} \tag{3.5}$$

which is a generalization of (1.6).

(a) If we take $\lambda = 0$ in (3.5), we get

$$\begin{aligned} I_{3,0}&(A:B)\\ &= \frac{1}{\alpha-\beta}\sum_{i=1}^{n}[\mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i)+(1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha}\mu_A^\beta(x_i)\mu_B^{1-\beta}(x_i)\\ &\qquad - (1-\mu_A(x_i))^\beta(1-\mu_B(x_i))^{1-\beta}] \end{aligned}$$

which of (1.6).

(b) If we take $\lambda = 0$ and $\beta = 1$ in (3.5), we get

$$I'_{3,0}(A:B) = \frac{1}{\alpha-1}\sum_{i=1}^{n}[\mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i)+(1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha}-1]$$

which is measure of fuzzy directed divergence corresponding to the probabilistic directed divergence defined by Havrda & Charvát [3].

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