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MATRIX SEMIRING

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ABSTRACT. In [6], we have recently proved that an additive inverse semiring S is a Clifford semifield if and only if S is a subdirect product of a field and a distributive lattice. In this paper, we study the matrix semiring over a Clifford semifield.

1. INTRODUCTION

Recall that a semiring $(S, +, \cdot)$ is a type $(2, 2)$ algebra whose semigroup reducts $(S, +)$ and $(S, \cdot)$ are connected by distributivity, that is, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in S$. We call a semiring $(S, +, \cdot)$ an additive inverse semiring if $(S, +)$ is an additive inverse semigroup. Additive inverse semirings were first studied by Karvellas [3] in 1974. In an additive inverse semiring $(S, +, \cdot)$, Karvellas [3] proved the following theorem.

Theorem 1.1. *Let S be an additive inverse semiring. Then for any $a, b \in S$ and $e \in E^+(S)$ we have (i) $(a')' = a$, (ii) $ab' = (ab)' = a'b$ (iii) $ab = a'b'$ and (iv) $e' = e$.*

An ideal I of a semiring S is a k-ideal of S if $a \in I$ and either $a + x \in I$ or $x + a \in I$ for some $x \in S$ implies $x \in I$. Also, an ideal I of a semiring S is called a full ideal if $E^+(S) \subseteq I$ where $E^+(S)$ denote the set of all additive idempotents of S.

Definition 1.2 ([8]). A semiring $(S, +, \cdot)$ is called a completely regular semiring if for every $a \in S$ there exists an element $x \in S$ such that

(i) $a + x + a = a$,
(ii) $a + x = x + a$ and
(iii) $a(a + x) = a + x$

It was proved in [8] that the condition (iii) can be replaced by the condition

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(iii)′ $(a+x)a = a+x$.

Definition 1.3 ([7]). A completely regular semiring S is a Clifford semiring if S is an additive inverse semiring such that $E^+(S)$ is a distributive sublattice of S as well as a k-ideal of S.

According to M. P. Grillet [2], a semiring $(S, +, \cdot)$ is called a skew-ring if its additive reduct $(S, +)$ is a group.

By using the concept of skew-ring, we proved the following theorem in [7].

Theorem 1.4. *A semiring S is a Clifford semiring if and only if S is a strong distributive lattice of skew-rings.*

By using Theorem 1.4, we see at once that if S is additive commutative then S is a Clifford semiring if and only if S is strong distributive lattice of rings.

Definition 1.5 ([6]). Let S be a Clifford semiring with 1 such that $1 \notin E^+(S)$. A non additive idempotent element $a \in S$ is said to be left invertible if there exists an element $r \in S$ such that $ra + 1 + 1' = 1$. In this case, r is called a left inverse of a. Similarly, we can define right invertible element in a Clifford semiring. An element is said to be invertible if it is left invertible as well as right invertible. If a is invertible, we say that a is a unit of S.

Definition 1.6 ([6]). A Clifford semiring S is called a Clifford semifield if the following conditions are satisfied :

(i) $1 \in S$ such that $1 \notin E^+(S)$,
(ii) S is both additive commutative and multiplicative commutative,
(iii) every non additive idempotent element of S is a unit.

Example 1.7. Let F be a field and D be a distributuve lattice with a greatest element 1_D. Then $F \times D$ is a Clifford semifield.

Definition 1.8. A full ideal I of a semiring S is called a minimal full ideal of S if there exists no ideal J of S such that $E^+(S) \subsetneq J \subsetneq I$.

Throughout this paper, S denotes a Clifford semifield with 0 and 1 and we denote an $n \times n$ matrix by $A = [\mathbf{a_1}, \ldots, \mathbf{a_i}, \ldots, \mathbf{a_n}]$, where $\mathbf{a_i}$ is the i-th column of the matrix A. Also, we write δ_i for the i-th column of I_n, the identity matrix.

Many aspects of the theory of matrices and determinants over semirings have been studied by Reutenauer and Straubing [4], Rutherford [5], Ghosh [1] and others. In this paper, we study some properties of determinants of square matrices over Clifford semifields with 0 and 1. Also, after introducing the concept of semi-invertibility of square matrices over Clifford semifields with 0 and 1, we obtain the necessary and sufficient condition for the semi-invertibility of square matrices. This paper ends with an application in solving a system of simultaneous linear equations over a Clifford semifield.

2. Determinant of Square Matrices

In this section, we study the determinant of a square matrix over a Clifford semifields. Throughout this paper, $M_n(S)$ denotes the set of all $n \times n$ square matrices over S. It can be easily verified that $M_n(S)$ is an additive inverse semiring but may not be a Clifford semiring.

Definition 2.1. A mapping $D : M_n(S) \longrightarrow S$ is said to be determinantal if it satisfies the properties

(2.1) $D[\ldots, \mathbf{b_i} + \mathbf{c_i}, \ldots] = D[\ldots, \mathbf{b_i}, \ldots] + D[\ldots, \mathbf{c_i}, \ldots]$;

(2.2) $D[\ldots, \lambda\mathbf{b_i}, \ldots] = \lambda D[\ldots, \mathbf{b_i}, \ldots]$;

(2.3) $D[\mathbf{a_1}, \ldots, \mathbf{a_i}, \ldots, \mathbf{a_j}, \ldots, \mathbf{a_n}] = (1')D[\mathbf{a_1}, \ldots, \mathbf{a_j}, \ldots, \mathbf{a_i}, \ldots, \mathbf{a_n}]$;

(2.4) $D[\delta_1, \ldots, \delta_i, \ldots, \delta_i, \ldots, \delta_n] = 0$;

(2.5) $D[\mathbf{a_1}, \ldots, \mathbf{a_{i-1}}, \mathbf{0}, \mathbf{a_{i+1}}, \ldots, \mathbf{a_n}] = 0$, where $\mathbf{0}$ denotes the column containing only 0;

(2.6) $D(I_n^*) = 1$ where I_n^* is an $n \times n$ matrix with diagonal elements 1 and all other elements are additive idempotents of S.

Theorem 2.2. *If D is a mapping that satisfies properties (2.1) and (2.3), then it satisfies the property* (2.3)′ *$D(A) \in E^+(S)$ whenever A has two identical columns.*

Proof. Taking $\mathbf{a_i} = \mathbf{a_j}$ with $i \neq j$ in (2.3), we obtain $D(A) = (D(A))'$. This leads to $2D(A) = D(A) + (D(A))'$. Since S is a Clifford semifield so that for $2 \cdot 1 \in S$, there exists an element $r \in S$ such that $2r + 1 + 1' = 1$. Now, $2D(A) = D(A) + (D(A))'$ implies $2rD(A) = rD(A) + r(D(A))'$. This leads to

$$\begin{aligned} 2rD(A) + D(A) + (D(A))' &= rD(A) + r(D(A))' + D(A) + (D(A))' \\ &= D(A) + (D(A))'. \end{aligned}$$

This implies $D(A) = D(A) + (D(A))'$. Hence, $D(A) \in E^+(S)$. □

If $A \in M_n(S)$ then we shall use the notation A_{ij} to denote the $(n-1)\times(n-1)$ matrix that is obtained from A by deleting the i-th row and the j-th column of A (i.e. the row and column containing a_{ij}). A_{ij} is called the minor of a_{ij} in A.

The following result shows how we can construct a determinantal mapping on the set of $n\times n$ matrices from a given determinantal mapping on the set of $(n-1)\times(n-1)$ matrices.

Theorem 2.3 *For $n>1$ let $D : M_{n-1}(S) \longrightarrow S$ be determinantal, and for $k=1,\dots,n$ define $f_k : M_n(S) \longrightarrow S$ by*

$$f_k(A)=\sum_{l=1}^{n}(1')^{k+l}a_{kl}D(A_{kl}).$$

Then each f_k is determinantal.

Proof. It is clear that $D(A_{kl})$ is independent of the l-th column of A and so $a_{kl}D(A_{kl})$ depends linearly on the l-th column of A. Consequently, we see that f_k depends linearly on the columns of A, i.e., f_k satisfies conditions (2.1) and (2.2) of the definition of a determinantal mapping.

We now show that f_k satisfies condition (2.3). Suppose that

$$A=[\mathbf{a_1},\dots,\mathbf{a_i},\dots,\mathbf{a_j},\dots,\mathbf{a_n}]$$

and

$$B=[\mathbf{b_1},\dots,\mathbf{b_i},\dots,\mathbf{b_j},\dots,\mathbf{b_n}]=[\mathbf{a_1},\dots,\mathbf{a_j},\dots,\mathbf{a_i},\dots,\mathbf{a_n}].$$

Then $b_{ki}=a_{kj}$ and $b_{kj}=a_{ki}$. Now for $l\neq i$ and $l\neq j$, A_{kl} and B_{kl} are two $(n-1)\times(n-1)$ matrices in which two columns are interchanged and so, since D is determinantal by hypothesis, we have

$$D(A_{kl})=(D(B_{kl}))'.$$

Suppose, without loss of generality, that $i<j$. Then it is clear that A_{ki} and A_{kj} can be transformed into B_{kj} and B_{ki} by effecting $(j-1-i)$ interchanges of adjacent columns; so, by property (2.3),

$$D(A_{ki})=(1')^{j-1-i}D(B_{kj})$$

and

$$D(A_{kj})=(1')^{j-1-i}D(B_{ki})$$

Since $a_{kl}=b_{kl}$ for all $l\neq i,j$ and $b_{ki}=a_{kj}, b_{kj}=a_{ki}$, we thus have

$$\begin{aligned}
f_k(A) &= \sum_{l=1}^{n}(1')^{k+l}a_{kl}D(A_{kl})\\
&= \sum_{l=1;l\neq i,j}^{n}(1')^{k+l}a_{kl}D(A_{kl})+(1')^{k+i}a_{ki}D(A_{ki})+(1')^{k+j}a_{kj}D(A_{kj})\\
&= \sum_{l=1;l\neq i,j}^{n}(1')^{k+l}(1')b_{kl}D(B_{kl})+(1')^{k+i}(1')^{j-1-i}b_{kj}D(B_{kj})\\
&\quad +(1')^{k+j}(1')^{j-1-i}b_{ki}D(B_{ki})\\
&= \sum_{l=1;l\neq i,j}^{n}(1')^{k+l}(1')b_{kl}D(B_{kl})+(1')^{k+j}(1')b_{kj}D(A_{kj})+(1')^{k+i}(1')b_{ki}D(B_{ki})\\
&= (1')\Bigg(\sum_{l=1;l\neq i,j}^{n}(1')^{k+l}b_{kl}D(B_{kl})+(1')^{k+j}b_{kj}D(A_{kj})+(1')^{k+i}b_{ki}D(B_{ki})\Bigg)\\
&= (1')\sum_{l=1}^{n}(1')^{k+l}b_{kl}D(B_{kl})\\
&= (1')f_k(B)
\end{aligned}$$

Hence $f_k([\mathbf{a_1},\ldots,\mathbf{a_i},\ldots,\mathbf{a_j},\ldots,\mathbf{a_n}])=(1')f_k([\mathbf{a_1},\ldots,\mathbf{a_j},\ldots,\mathbf{a_i},\ldots,\mathbf{a_n}])$

One can easily show that f_k satisfies property (2.4) and (2.5).

Finally, f_k satisfies property (2.6) since if $A=I_n^*$ then $a_{kk}=1$ and $a_{kl}=e$ for some $e\in E^+(S)$ where $k\neq l$ and $A_{kk}=I_{n-1}^*$, so that

$$f_k(I_n^*)=(1')^{k+k}D(I_{n-1}^*)+f\ (\text{ where} f\in E^+(S))\ =1.$$

Consequently, it follows that f_k is determinantal for every k. □

Corollary 2.4. *For every positive integer n there is atleast one determinantal mapping on $M_n(S)$.*

Proof. We prove it by induction. The result is trivial for $n=1$ and Theorem 2.3 shows how atleast one such mapping can be defined on $M_n(S)$ from a given determinantal mapping on $M_{n-1}(S)$. □

Definition 2.5. Let σ be a pertutation on the set $\{1,2,\ldots,n\}$. Define ϵ_σ by

$$\epsilon_\sigma=\begin{cases}1 & \text{if } \sigma \text{ is an even permutation}\\ 1' & \text{if } \sigma \text{ is an odd permutation}\end{cases}$$

Then by condition (2.3) and Definition 2.5, we at once get

$$D[a_{\sigma(1)},a_{\sigma(2)},\ldots,a_{\sigma(n)}=\epsilon_\sigma D[a_1,a_2,\ldots,a_n]$$

Proceeding as in the case of determinant over a field, we can prove the following theorem.

Theorem 2.6. *There is one and only one determinantal mapping $D : M_n(S) \longrightarrow S$ and it can be described by*

$$D(A) = \sum_{\sigma \in S_n} \epsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n),n}.$$

An important consequence of the above result is that the expression for $f_k(A)$ given in Theorem 2.3 is independent of k.

Definition 2.7. The unique determinantal mapping on $M_n(S)$ will be denoted by det. By the determinant of $A = [a_{ij}]_{n \times n}$ we shall mean det$A$.

By Theorem 2.6, we see that $\det A = \sum_{\sigma \in S_n} \epsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n),n}$.

Alternatively, by Theorem 2.3, we have $\det A = \sum_{j=1}^{n} (1')^{i+j} a_{ij} \det A_{ij}$, which will be called the Laplace expansion along the i-th row. It is noteworthy that the Laplace expansion is independent of the row chosen.

For a semiring S and a matrix $A \in M_n(S)$, Reutenauer and Straubing [4] have defined the positive determinant $|A|^+$ and negative determinant $|A|^-$ as follows :

$$|A|^+ = \sum_{\sigma \in A_n} a_{\sigma(1),1} \cdots a_{\sigma(n),n} \text{ and } |A|^- = \sum_{\sigma \in S_n \setminus A_n} a_{\sigma(1),1} \cdots a_{\sigma(n),n}.$$

By the help of $|A|^+$ and $|A|^-$ we at once have

$$\begin{aligned}
detA &= \sum_{\sigma \in S_n} \epsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n),n} \\
&= \sum_{\sigma \in A_n} \epsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n),n} + \sum_{\sigma \in S_n \setminus A_n} \epsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n),n} \\
&= \sum_{\sigma \in A_n} a_{\sigma(1),1} \cdots a_{\sigma(n),n} + \sum_{\sigma \in S_n \setminus A_n} 1' a_{\sigma(1),1} \cdots a_{\sigma(n),n} \\
&= \sum_{\sigma \in A_n} a_{\sigma(1),1} \cdots a_{\sigma(n),n} + (1') \sum_{\sigma \in S_n \setminus A_n} a_{\sigma(1),1} \cdots a_{\sigma(n),n} \\
&= (|A|^+) + (1')(|A|^-) \\
&= (|A|^+) + (|A|^-)'
\end{aligned}$$

We can prove the following theorem as in the case of determinant over a field.

Theorem 2.8. *For a square matrix* $A = [a_{ij}]_{n\times n}$, $detA = detA^t$.

Corollary 2.9. $(j = 1, \ldots, n)$ $detA = \sum_{i=1}^{n} (1')^{i+j} a_{ij}\ detA_{ij}$.

Theorem 2.10. *For a square matrix* $A = [a_{ij}]_{n\times n}$, $det[I_n^* + (I_n^*)' + A] = 1 + 1' + detA$.

Proof. Let $B = [b_{ij}]_{n\times n} = I_n^* + (I_n^*)' + A$. Now, we have $b_{ii} = 1 + 1' + a_{ii}$ and $b_{ij} = e_{ij} + a_{ij}$ for $i \neq j$, where $e_{ij} \in E^+(S)$. Then

$$\begin{aligned}
detB &= \sum_{\sigma\in S_n} \epsilon_\sigma b_{\sigma(1),1} \cdots b_{\sigma(n),n} \\
&= b_{11} \ldots b_{nn} + \sum_{\sigma\in S_n\setminus\{id\}} \epsilon_\sigma b_{\sigma(1),1} \cdots b_{\sigma(n),n} \\
&= ((1 + 1' + a_{11}) \ldots (1 + 1' + a_{nn})) + \sum_{\sigma\in S_n\setminus\{id\}} \epsilon_\sigma (e_{\sigma(1),1} \\
&\quad + a_{\sigma(1),1}) \cdots (e_{\sigma(n),n} + a_{\sigma(n),n}) \\
&\qquad\qquad [\text{where } id \text{ denotes the identity permutation}] \\
&= 1 + 1' + a_{11} \ldots a_{nn} + e + \sum_{\sigma\in S_n\setminus\{id\}} \epsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n),n}, \text{ for some } e \in E^+(S) \\
&= 1 + 1' + \sum_{\sigma\in S_n} \epsilon_\sigma a_{\sigma(1),1} \cdots a_{\sigma(n),n} \\
&= 1 + 1' + detA
\end{aligned}$$

Therefore, $\det[I_n^* + (I_n^*)' + A] = 1 + 1' + \det A$. □

Theorem 2.11. *If* $A = [a_{ij}]$ *and* $B = [b_{ij}]$ *are two* $n \times n$ *matrices over* S *then* $det(AB) = (detA)(detB) + e$ *for some* $e \in E^+(S)$.

Proof. If $C = AB$ then the k-th column of C can be written $\mathbf{c_k} = b_{1k}\mathbf{a_1} + \ldots + b_{nk}\mathbf{a_n}$. Moreover the i-th entry of $\mathbf{c_k}$ is $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$.

Thus we have

$$\begin{aligned}
det(AB) &= detC \\
&= det[b_{11}\mathbf{a_1} + \ldots + b_{n1}\mathbf{a_n} + \ldots + b_{1n}\mathbf{a_1} + \ldots + b_{nn}\mathbf{a_n}]
\end{aligned}$$

By using property (2.1) in the definition of the determinant we can write $\det(AB)$ as a sum of the terms of the form

$$det[b_{\sigma(1),1}\mathbf{a_{\sigma(1)}}, \ldots\ldots, b_{\sigma(n),n}\mathbf{a_{\sigma(n)}}],$$

where $1 \leq \sigma(i) \leq n$ for every i. Using property (7.1.2), we can express each of these terms as $b_{\sigma(1),1} \cdots\cdots, b_{\sigma(n),n}$ $\det[\mathbf{a}_{\sigma(1)}, \ldots, \mathbf{a}_{\sigma(\mathbf{n})}]$.
However by (2.3)$'$ each expression is in $E^+(S)$ except for those in which $\sigma(i) \neq \sigma(j)$ for $i \neq j$; i.e., those in which σ is a permutation on $\{1, \ldots, n\}$. Thus we can deduce that

$$\begin{aligned}
det(AB) &= det[b_{11}\mathbf{a_1} + \ldots + b_{n1}\mathbf{a_n} + \ldots + b_{1n}\mathbf{a_1} + \ldots + b_{nn}\mathbf{a_n}] \\
&= \Big(\sum_{\sigma \in S_n} det[b_{\sigma(1),1}\mathbf{a}_{\sigma(1)}, \ldots\ldots, b_{\sigma(n),n}\mathbf{a}_{\sigma(\mathbf{n})}]\Big) + e \text{ for some } e \in E^+(S) \\
&= \Big(\sum_{\sigma \in S_n} b_{\sigma(1),1} \cdots\cdots b_{\sigma(n),n} det[\mathbf{a}_{\sigma(1)}, \ldots, \mathbf{a}_{\sigma(\mathbf{n})}]\Big) + e \\
&= \Big(\sum_{\sigma \in S_n} b_{\sigma(1),1} \cdots\cdots b_{\sigma(n),n} \epsilon_\sigma det[\mathbf{a_1}, \ldots, \mathbf{a_n}]\Big) + e \\
&= (detA)\Big(\sum_{\sigma \in S_n} \epsilon_\sigma b_{\sigma(1),1} \cdots\cdots b_{\sigma(n),n}\Big) + e \\
&= (detA)(detB) + e.
\end{aligned}$$

Hence the result follows. □

It is to be noted that we can also prove the above theorem by using Lemma 1 of [4]. The proof is as follows :

$$|AB|^+ = (|A|^+)(|B|^+) + (|A|^-)(|B|^-) + r$$

and

$$|AB|^- = (|A|^+)(|B|^-) + (|A|^-)(|B|^+) + r$$

for some $r \in S$.

$$\begin{aligned}
det(AB) &= (|AB|^+) + (|AB|^-)' \\
&= (|A|^+)(|B|^+) + (|A|^-)(|B|^-) + r + ((|A|^+)(|B|^-) + (|A|^-)(|B|^+) + r)' \\
&= (|A|^+)(|B|^+) + ((|A|^+)(|B|^-))' + (|A|^-)(|B|^-) + ((|A|^-)(|B|^+))' + r \\
&\quad + r' \\
&= (|A|^+)((|B|^+) + (|B|^-))') + (|A|^-)((|B|^+) + (|B|^-))') + r + r' \\
&= (|A|^+)detB + (|A|^-)detB + r + r' \\
&= (detB)((|A|^+) + (|A|^-)) + e \text{ where } e = r + r' \\
&= (detA)(detB) + e
\end{aligned}$$

Corollary 2.12. *For two matrices $A, B \in M_n(S)$, $det(AB) \notin E^+(S)$ if and only if $detA \notin E^+(S)$ and $detB \notin E^+(S)$.*

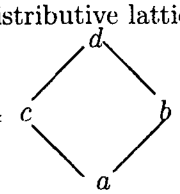
3. Semi-Invertibility of Square Matrices

An $n \times n$ matrix A over a semiring S is said to be invertible if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. In this section, we define semi-invertibility of a square matrix over a Clifford semifield S with 0 and 1 and obtain the necessary and sufficient condition for semi-invertibility of square matrices over S. Throughout this section, let $E^+(M_n(S))$ be the set of all $n \times n$ matrices whose all entries are additive idempotents and $I^+(M_n(S))$ denote the set of all $n \times n$ matrices whose (i, i)-th entries are 1 and all other entries are additive idempotents. It can be easily verified that $E^+(M_n(S))$ is a k-ideal of $M_n(S)$.

Definition 3.1. An $n \times n$ matrix A over S is said to be right semi-invertible if there exists $B \in M_n(S)$ such that $I_n^* + (I_n^*)' + AB = I_n^{**}$ for some $I_n^*, I_n^{**} \in I^+(M_n(S))$. Similarly, we can define left semi-invertiblity of a square matrix. A square matrix is said to be semi-invertible if it is left semi-invertible as well as right semi-invertible.

From definition, it follows that if a matrix $A \in M_n(S)$ is invertible then it is semi-invertible. But the converse may not be true. This follows from the following example.

Example 3.2. Let D be a distributive lattice given by

D =


Let $S = (\mathbb{Q} \times \{b, d\}) \cup (\mathbb{R} \times \{a, c\})$, where $\mathbb{R}$ and $\mathbb{Q}$ are the fields of real numbers and rational numbers respectively. Then S is a Clifford semifield with $(0, a)$ as its zero element and $(1, d)$ as its identity. Now, on S, we consider the matrix

$$A = \begin{pmatrix} (1, a) & (3, a) \\ (2, a) & (5, a) \end{pmatrix}.$$

We show that for the matrix A there is no matrix $B \in M_2(S)$ such that $AB = BA = I_2$. If possible there exist a matrix $B = \begin{pmatrix} (x,\alpha) & (y,\beta) \\ (z,\gamma) & (u,\delta) \end{pmatrix} \in M_2(S)$ such that $AB = BA = I_2$. Then from $AB = I_2$ we have immediately

$$\begin{pmatrix} (x+3z,a) & (y+3u,a) \\ (2x+5z,a) & (2y+5u,a) \end{pmatrix} = \begin{pmatrix} (1,d) & (0,a) \\ (0,a) & (1,d) \end{pmatrix}.$$

Comparing the corresponding entries, we have $a = d$, a contradiction. Thus, there does not exit no such a matrix $B \in M_2(S)$ such that $AB = BA = I_2$. Now

$$C = \begin{pmatrix} (-5,a) & (3,a) \\ (2,a) & (-1,a) \end{pmatrix} \in M_2(S)$$

such that

$$I_2 + I_2' + CA = \begin{pmatrix} (1,d) & (0,a) \\ (0,a) & (1,d) \end{pmatrix} = I_2.$$

Thus A is clearly left semi-invertible. Similarly, we can show that A is right semi-invertible.

Theorem 3.3. *If an $n \times n$ matrix A over S is right (left) semi-invertible then $detA \notin E^+(S)$.*

Proof. Since A is right semi-invertible, there exists an $n \times n$ matrix B such that $I_n^* + (I_n^*)' + AB = I_n^{**}$ for some $I_n^*, I_n^{**} \in I^+(M_n(S))$. This implies $\det(I_n^* + (I_n^*)' + AB) = \det(I_n^{**}) = 1$. Then by the above results and Theorem 2.10, we can easily deduce that $1 + 1' + \det(AB) = 1$. Again by Theorem 2.11, we have $\det(AB) = (\det A)(\det B) + e$, for some $e \in E^+(S)$. Hence $1 + 1' + (\det A)(\det B) = 1$. This leads to $\det A \notin E^+(S)$. □

Theorem 3.4. *If $detA \notin E^+(S)$ for an $n \times n$ matrix A over S then A is right (left) semi-invertible.*

Proof. Let $A = [a_{ij}]_{n\times n} \in M_n(S)$ be such that $\det A \notin E^+(S)$. Then there exists an element $r \in S$ such that $1 + 1' + r\det A = 1$. Let $B = [b_{ij}]_{n\times n} = [r(1')^{i+j}\det A_{ij}]^t_{n\times n}$. Let $I_n^* \in I^+(M_n(S))$ and $I_n^* + (I_n^*)' + AB = C = [c_{ij}]_{n\times n}$. Then

$$\begin{aligned} c_{ii} &= 1 + 1' + \sum_{j=1}^{n} a_{ij} b_{ji} \\ &= 1 + 1' + \sum_{j=1}^{n} a_{ij} r (1')^{i+j} \det A_{ij} \end{aligned}$$

$$= 1 + 1' + r\Big(\sum_{j=1}^{n}(1')^{i+j}a_{ij}\det A_{ij}\Big)$$
$$= 1 + 1' + rdetA$$
$$= 1.$$

Again for $i \neq j$

$$c_{ij} = [I_n^*]_{ij} + ([I_n^*])'_{ij} + \sum_{k=1}^{n} a_{ik}b_{kj}$$
$$= [I_n^*]_{ij} + ([I_n^*])'_{ij} + \sum_{j=1}^{n} a_{ik}r(1')^{k+j}\det A_{jk}$$
$$= [I_n^*]_{ij} + ([I_n^*])'_{ij} + r\Big(\sum_{j=1}^{n}(1')^{i+j}a_{ik}\det A_{jk}\Big)$$
$$= [I_n^*]_{ij} + ([I_n^*])'_{ij} + re \text{ (for some } e \in E^+(S)) \in E^+(S).$$

Hence $I_n^* + (I_n^*)' + AB = I_n^{**}$ for some $I_n^{**} \in I^+(M_n(S))$. Consequently, A is right (left) semi-invertible. □

Corollary 3.5. *An $n \times n$ matrix A over S is right (left) semi-invertible if and only if $detA \notin E^+(S)$.*

Lemma 3.6. *Let $I_n^* \in I^+(M_n(S))$ and $A \in M_n(S)$. Then $I_n^* A = A + O_{E^+}^*$ and $A I_n^* = A + O_{E^+}^{**}$ for some $O_{E^+}^*, O_{E^+}^{**} \in E^+(M_n(S))$.*

Proof. Now $I_n^* = I_n + O_{E^+}^{*1}$ for some $O_{E^+}^{*1} \in E^+(M_n(S))$. Hence $A I_n^* = A(I_n + O_{E^+}^{*1}) = AI_n + A O_{E^+}^{*1} = A + O_{E^+}^*$ for some $O_{E^+}^* \in E^+(M_n(S))$.

Similarly, we can show that $A I_n^* = A + O_{E^+}^{**}$ for some $O_{E^+}^{**} \in E^+(M_n(S))$. □

Theorem 3.7. *For an $n \times n$ matrix A over S if $I_n^* + (I_n^*)' + AB = I_n^{**}$ for some $B \in M_n(S)$ and $I_n^*, I_n^{**} \in I^+(M_n(S))$ then $I_n^{*1} + (I_n^{*1})' + BA = I_n^{***}$ for some $I_n^{*1}, I_n^{***} \in I^+(M_n(S))$.*

Proof. Now, $I_n^* + (I_n^*)' + AB = I_n^{**}$ implies $\det[I_n^* + (I_n^*)' + AB] = \det(I_n^{**})$. Then by Theorem 2.10, it follows that $1 + 1' + \det(AB) = 1$. Again applying Theorem 2.11, we have $1 + 1' + \det(A)\det(B) = 1.4$. This leads to $\det B \notin E^+(S)$. Hence by Theorem 3.4, B is right semi-invertible. So there exists a matrix $C \in M_n(S)$ such that $I_n^{*2} + (I_n^{*2})' + BC = I_n^{*3}$. Now $I_n^* + (I_n^*)' + AB = I_n^{**}$ implies $B(I_n^* +$

$(I_n^*)'C + B(AB)C = B(I_n^{**})C$. This leads to $BI_n^*C + B(I_n^*)'C + BABC = BI_n^{**}C$. Then by Lemma 3.6, $BC + (BC)' + O_{E^+}^* + BABC = O_{E^+}^{*1} + BC$. This implies $I_n^{*2} + (I_n^{*2})' + BC + (BC)' + O_{E^+}^* + BABC = O_{E^+}^{*1} + I_n^{*2} + (I_n^{*2})' + BC$. This leads to, $I_n^{*3} + (I_n^{*3})' + O_{E^+}^* + BABC = O_{E^+}^{*1} + I_n^{*3} = I_n^{*4}$ where $O_{E^+}^{*1} + I_n^{*3} = I_n^{*4}$. This implies $I_n^{*3} + (I_n^{*3})' + O_{E^+}^* + BABC + BAI_n^{*2} + BA(I_n^{*2})' = I_n^{*4} + BAI_n^{*2} + BA(I_n^{*2})'$, i.e., $I_n^{*3} + (I_n^{*3})' + O_{E^+}^* + BA(I_n^{*2} + (I_n^{*2})' + BC) = I_n^{*4} + BAI_n^{*2} + BA(I_n^{*2})'$. This implies $I_n^{*3} + (I_n^{*3})' + O_{E^+}^* + BAI_n^{*3} = I_n^{*4} + BAI_n^{*2} + BA(I_n^{*2})'$. Then by Lemma 3.6, it follows that $I_n^{*3} + (I_n^{*3})' + O_{E^+}^* + BA + 0_{E^+}^{*2} = I_n^{*4} + BAI_n^{*2} + BA(I_n^{*2})'$. This implies $I_n^{*1} + (I_n^{*1})' + BA = I_n^{***}$ where $I_n^{*1} + (I_n^{*1})' = I_n^{*3} + (I_n^{*3})' + O_{E^+}^* + 0_{E^+}^{*2}$ and $I_n^{***} = I_n^{*4} + BAI_n^{*2} + BA(I_n^{*2})'$. Hence the theorem is complete. □

Corollary 3.8. *An $n \times n$ matrix A over S is left semi-invertible if and only if it is right semi-invertible.*

Proof. Follows from Theorem 3.7.

Definition 3.9. For a matrix $A \in M_n(S)$ if there exists a matrix $B \in M_n(S)$ such that $I_n^* + (I_n^*)' + AB = I_n^{**}$ for some $I_n^*, I_n^{**} \in I^+(M_n(S))$ then B is called a left semi-inverse of the matrix A. Similarly, we can define the right semi-inverse of a matrix over S.

By Theorem 3.7, we at once have $B \in M_n(S)$ is a left semi-inverse of $A \in M_n(S)$ if and only if B is right semi-inverse of A. In this case, B is called a semi-inverse of A.

Similar to Clifford semifield, it is important to note that the semi-inverse of a matrix A with $\det A \notin E^+(S)$ may not be unique. For this purpose we consider the following example.

Example 3.10. We consider the Clifford semifield S in Example 3.2. On S, we consider the matrix

$$A = \begin{pmatrix} (\sqrt{2}, a) & (1, c) & (2, a) \\ (2\sqrt{2}, a) & (-2, b) & (-5, a) \\ (1, a) & (\sqrt{2}, c) & (-\sqrt{2}, a) \end{pmatrix}.$$

Let

$$C = \begin{pmatrix} (\frac{7\sqrt{2}}{25}, a) & (\frac{3\sqrt{2}}{25}, a) & (-\frac{1}{25}, a) \\ (-\frac{1}{25}, a) & (-\frac{4}{25}, b) & (\frac{9\sqrt{2}}{25}, a) \\ (\frac{6}{25}, a) & (-\frac{1}{25}, a) & (-\frac{4\sqrt{2}}{25}, a) \end{pmatrix}.$$

Then $I_3 + I_3' + CA = I_3$. Hence C is a left semi-inverse of A.

Again, let

$$C_1 \doteq \begin{pmatrix} (\frac{7\sqrt{2}}{25}, c) & (\frac{3\sqrt{2}}{25}, a) & (-\frac{1}{25}, b) \\ (-\frac{1}{25}, a) & (-\frac{4}{25}, d) & (\frac{9\sqrt{2}}{25}, c) \\ (\frac{6}{25}, b) & (-\frac{1}{25}, d) & (-\frac{4\sqrt{2}}{25}, c) \end{pmatrix}.$$

Then

$$\begin{pmatrix} (1, d) & (0, b) & (o, c) \\ (0, a) & (1, d) & (0, b) \\ (0, c) & (0, d) & (1, d) \end{pmatrix} + \begin{pmatrix} (-1, d) & (0, b) & (o, c) \\ (0, a) & (-1, d) & (0, b) \\ (0, c) & (0, d) & (-1, d) \end{pmatrix} + C_1 A$$

$$= \begin{pmatrix} (1, d) & (0, d) & (0, c) \\ (0, a) & (1, d) & (0, b) \\ (0, c) & (0, d) & (1, d) \end{pmatrix} \in I^+(M_n(S))$$

Thus, C_1 is also a semi-inverse of A. This shows that semi-inverse of a matrix over a Clifford semifield is not unique.

Definition 3.11. For $A \in M_n(S)$, the set of all semi-inverses are denoted by $V(A)$ and is defined by

$$V(A) = \{B \in M_n(S) : I_n^* + (I_n^*)' + AB = I_n^{**} \text{ for some } I_n^*,\ I_n^{**} \in I^+(M_n(S))\}.$$

Theorem 3.12. *Let $A \in M_n(S)$ be such that $detA \notin E^+(S)$. Then $(V(A))^t = V(A^t)$.*

Proof. Let $B \in (V(A))^t$. Then $B^t \in V(A)$. So we have

$$I_n^* + (I_n^*)' + AB^t = I_n^{**} \text{ for some } I_n^*,\ I_n^{**} \in I^+(M_n(S))$$
$$\text{i.e., } (I_n^* + (I_n^*)' + AB^t)^t = (I_n^{**})^t$$
$$\text{i.e., } I_n^{*1} + (I_n^{*1})' + (AB^t)^t = I_n^{*2} \text{ where } (I_n^*)^t = I_n^{*1} \text{ and } (I_n^{**})^t = I_n^{*2}$$
$$\text{i.e., } I_n^{*1} + (I_n^{*1})' + BA^t = I_n^{*2}$$
$$\text{i.e., } I_n^{*3} + (I_n^{*3})' + A^tB = I_n^{*4} \text{ for some } I_n^{*3},\ I_n^{*4} \in I^+(M_n(S)).$$

Hence we have $B \in V(A^t)$ and thus $(V(A))^t \subseteq V(A^t)$.

Again let $B \in V(A^t)$. Then we have

$$I_n^* + (I_n^*)' + A^tB = I_n^{**} \text{ for some } I_n^*,\ I_n^{**} \in I^+(M_n(S))$$
$$\text{i.e., } (I_n^* + (I_n^*)' + A^tB)^t = (I_n^{**})^t$$
$$\text{i.e., } I_n^{*1} + (I_n^{*1})' + (A^tB)^t = I_n^{*2} \text{ where } (I_n^*)^t = I_n^{*1} \text{ and } (I_n^{**})^t = I_n^{*2}$$
$$\text{i.e., } I_n^{*1} + (I_n^{*1})' + B^tA = I_n^{*2}$$
$$\text{i.e., } I_n^{*3} + (I_n^{*3})' + AB^t = I_n^{*4} \text{ for some } I_n^{*3}, I_n^{*4} \in I^+(M_n(S)).$$

This gives $B^t \in V(A)$. This leads to $B \in (V(A))^t$. Hence $V(A^t) \subseteq (V(A))^t$. Consequently, $(V(A))^t = V(A^t)$. □

Theorem 3.13. *Let A, $B \in M_n(S)$ be two semi-invertible matrices. Then*

$$V(B)V(A) \subseteq V(AB).$$

Proof. Let $A^* \in V(A)$ and $B^* \in V(B)$. Then there exist matrices $I_n^*, I_n^{**}, I_n^{*1}, I_n^{*2} \in I^+(M_n(S))$ such that $I_n^* + (I_n^*)' + A^*A = I_n^{**}$ and $I_n^{*1} + (I_n^{*1})' + B^*B = I_n^{*2}$. Now we can deduce that

$I_n^* + (I_n^*)' + A^*A = I_n^{**}$

i.e., $B^*\big(I_n^* + (I_n^*)' + A^*A\big)B = B^*I_n^{**}B$

i.e., $B^*I_n^*B + (B^*I_n^*B)' + B^*A^*AB = B^*I_n^{**}B$

i.e., $0_n^* + B^*A^*AB = B^*(I_n + 0_n^{**})B$

[where $0_n^* = B^*I_n^*B + (B^*I_n^*B)' \in E^+(M_n(S))$ and $I_n^{**} = I_n + 0_n^{**} \in I^+(M_n(S))$]

i.e., $0_n^* + B^*A^*AB = B^*B + B^*0_n^{**}B$

i.e., $I_n^{*1} + (I_n^{*1})' + 0_n^* + B^*A^*AB = I_n^{*1} + (I_n^{*1})' + B^*B + B^*0_n^{**}B$

i.e., $(I_n^{*1} + 0_n^*) + (I_n^{*1} + 0_n^*)' + B^*A^*AB = I_n^{*2} + B^*0_n^{**}B$

i.e., $I_n^{*3} + (I_n^{*3})' + B^*A^*AB = I_n^{*4}$

[where $I_n^{*3} = I_n^{*1} + 0_n^*$ and $I_n^{*4} = I_n^{*2} + B^*0_n^{**}B$]

Hence, $B^*A^* \in V(AB)$. Thus, $V(B)V(A) \subseteq V(AB)$. □

Application

Problem 3.14. We consider the Clifford semifield S in Example 3.2. Solve the following system of equations :

$$(\sqrt{2}, a)x + (1, c)y + (2, a)z = (3, a)$$
$$(2\sqrt{2}, a)x + (-2, b)y + (-5, a)z = (4, b)$$
$$(1, a)x + (\sqrt{2}, c)y + (-\sqrt{2}, a)z = (8\sqrt{2}, a)$$

where x, y, $z \in S$.

Solution. Let

$$A = \begin{pmatrix} (\sqrt{2}, a) & (1, c) & (2, a) \\ (2\sqrt{2}, a) & (-2, b) & (-5, a) \\ (1, a) & (\sqrt{2}, c) & (-\sqrt{2}, a) \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and

$$B = \begin{pmatrix} (3, a) \\ (4, b) \\ (8\sqrt{2}, a) \end{pmatrix},$$

where $x = (x_{\mathbb{R}},\ x_D)$, $y = (y_{\mathbb{R}},\ y_D)$ and $z = (z_{\mathbb{R}},\ z_D)$.
Then the given system of equations can be written as $AX = B$.

Now,

$$\begin{aligned} detA &= (\sqrt{2}, a)(7\sqrt{2}, a) + (-1, c)(1, a) + (2, a)(6, a) \\ &= (14, a) + (-1, a) + (12, a) \\ &= (25, a) \notin E^+(S). \end{aligned}$$

Hence, by Theorem 3.4, A is semi-invertible. Let

$$C = \begin{pmatrix} (\frac{7\sqrt{2}}{25}, a) & (\frac{3\sqrt{2}}{25}, a) & (-\frac{1}{25}, a) \\ (-\frac{1}{25}, a) & (-\frac{4}{25}, b) & (\frac{9\sqrt{2}}{25}, a) \\ (\frac{6}{25}, a) & (-\frac{1}{25}, a) & (-\frac{4\sqrt{2}}{25}, a) \end{pmatrix}.$$

Now,

$$I_3 + I_3' + CA$$

$$= I_3 + I_3' + \begin{pmatrix} (\frac{7\sqrt{2}}{25}, a) & (\frac{3\sqrt{2}}{25}, a) & (-\frac{1}{25}, a) \\ (-\frac{1}{25}, a) & (-\frac{4}{25}, b) & (\frac{9\sqrt{2}}{25}, a) \\ (\frac{6}{25}, a) & (-\frac{1}{25}, a) & (-\frac{4\sqrt{2}}{25}, a) \end{pmatrix} \begin{pmatrix} (\sqrt{2}, a) & (1, c) & (2, a) \\ (2\sqrt{2}, a) & (-2, b) & (-5, a) \\ (1, a) & (\sqrt{2}, c) & (-\sqrt{2}, a) \end{pmatrix}$$

$$= \begin{pmatrix} (1, d) & (0, a) & (0, a) \\ (0, a) & (1, d) & (0, a) \\ (0, a) & (0, a) & (1, d) \end{pmatrix} = I_3$$

This implies

$$(I_3 + I_3' + CA)X = I_3X$$

This leads to

$$X + X' + CAX = X$$

This implies

$$X + X' + CB = X.$$

This leads to

$$\begin{pmatrix} (0, x_D) \\ (0, y_D) \\ (0, z_D) \end{pmatrix} + \begin{pmatrix} (\sqrt{2}, a) \\ (5, b) \\ (-2, a) \end{pmatrix} = \begin{pmatrix} (x_{\mathbb{R}}, x_D) \\ (y_{\mathbb{R}}, y_D) \\ (z_{\mathbb{R}}, z_D) \end{pmatrix}.$$

This implies

$$\begin{pmatrix} (\sqrt{2}, a + x_D) \\ (5, b + y_D) \\ (-2, a + z_D) \end{pmatrix} = \begin{pmatrix} (x_{\mathbb{R}}, x_D) \\ (y_{\mathbb{R}}, y_D) \\ (z_{\mathbb{R}}, z_D) \end{pmatrix}.$$

This leads to

$$x_{\mathbb{R}} = \sqrt{2},\ y_{\mathbb{R}} = 5,\ z_{\mathbb{R}} = -2,\ a + x_D = x_D,\ b + y_D = y_D \text{ and } a + z_D = z_D.$$

Hence we have

$$x_{\mathbb{R}} = \sqrt{2},\ y_{\mathbb{R}} = 5,\ z_{\mathbb{R}} = -2,\ x_D \geq a,\ y_D \geq b,\ z_D \geq a.$$

Thus,

$$x = (x_{\mathbb{R}}, x_D) = \{\sqrt{2}\} \times \{a, c\},\ y = (y_{\mathbb{R}}, y_D) = \{5\} \times \{b, d\}$$

and

$$z = (z_{\mathbb{R}}, z_D) = \{-2\} \times D.$$

Consequently, the set of solution of the given system of equations is

$$x = \{\sqrt{2}\} \times \{a, c\},\ y = \{5\} \times \{b, d\},\ z = \{-2\} \times D.$$

□

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