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P(R, M) GAMMA NEAR-RINGS

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ABSTRACT. In this paper, we introduce the concept of P(r,m) Γ- near-ring and obtain some characterization of P(r,m) Γ-near-rings through regularity conditions.

1. INTRODUCTION

Throughout this paper M stands for a Γ-near-ring. For basic definitions in near-ring theory one may refer to Pilz [3] and in Γ-near-ring one may refer to [4].The concept of P(r,m) near-rings was introduced by Balakrishnan [1]. Now we introduce the concept of P(r,m) Γ-near-rings and obtain some characterization of the same through regularity conditions. Further we obtain some properties of P(1,2) and P(2,1) Γ-near-rings.

2. PRELIMINARIES

A Γ-near-ring is a triple $(M, +, \Gamma)$ where

(i) $(M, +)$ is a group,

(ii) Γ is a non-empty set of binary operations on M such that for each $\gamma \in \Gamma$, $(M, +, \gamma)$ is a right near-ring,

(iii) $x\gamma(y\mu z) = (x\gamma y)\mu z$ for all $x, y, z \in M$ and $\gamma, \mu \in \Gamma$.

For $x \in M$ and a positive integer m, by x^m we mean $x\gamma_1 x\gamma_2 ... x\gamma_{m-1} x$, where $\gamma_i \in \Gamma$ for $1 \leq i \leq m-1$. M is said to be a $P(r, m)$ Γ-near-ring if there exist positive integers r, m such that $x^r\Gamma M = M\Gamma x^m$ for all $x \in M$. M is called a left unital Γ-near-ring (right unital Γ-near-ring) if $x \in x\Gamma M$ ($x \in M\Gamma x$) for all $x \in M$. M is said to be regular if for each $a \in M$, there exists $b \in M$ such that $a = a\gamma_1 b\gamma_2 a$ for

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every pair of non-zero elements γ_1, γ_2 of Γ. A non-empty subset A of M is called left Γ-subgroup (right Γ-subgroup) of M if A is a subgroup of (M,+) and $M\Gamma A \subseteq A$ ($A\Gamma M \subseteq A$).

M is said to fulfill the intertion of factors property (IFP) provided that for all $a, b \in M$, $a\gamma b = 0$ for all $\gamma \in \Gamma$ implies $a\gamma_1 n\gamma_2 b = 0$ for every pair of non-zero elements γ_1, γ_2 of Γ and for all $n \in M$. A Γ-near-ring M is said to be zero symmetric if $x\gamma 0 = 0$ for all $x \in M$ and $\gamma \in \Gamma$. A Γ-near-ring M is said to be sub-commutative if $x\gamma_1 M = M\gamma_2 x$ for all $x \in M$ and $\gamma_1, \gamma_2 \in \Gamma$.

M is said to be left permutable or right permutable [2] according as $(x\gamma_1 y)\gamma_2 z = (y\gamma_1 x)\gamma_2 z$ or $(x\gamma_1 y)\gamma_2 z = (x\gamma_1 z)\gamma_2 y$ for all $x, y, z \in M$ and for every pair of non-zero elements γ_1, γ_2 of Γ. For $A \subseteq M$, we define the radical $\sqrt{A}$ of A to be $\{x \in M / x^k \in A$ for some positive integer $k\}$. A regular Γ-near-ring M is called P_3 regular if for each $a \in M, a\gamma b = b\gamma a$ for all $\gamma \in \Gamma$, where b is an element M satisfying the property $a = a\gamma_1 b\gamma_2 a$ for every pair of non-zero elements γ_1, γ_2 of Γ.

M is said to have strong IFP if for all ideals I of M and for all $a, b, n \in M, a\gamma b \in I$ for all $\gamma \in \Gamma$ implies $a\gamma_1 n\gamma_2 \in I$ for every pair of non-zero elements γ_1, γ_2 of Γ. M is said to be a generalized gamma-near-field (GGNF) if for each $a \in M$, there exists a unique $a' \in M$ such that $a\gamma_1 a'\gamma_2 a = a$ and $a'\gamma_1 a\gamma_2 a' = a'$ for every pair of non-zero elements γ_1, γ_2 of Γ [5]. It is easy to see that a P(r,m) Γ-near-ring is zero-symmetric. An element $a \in M$ is called idempotent if $a\gamma a = a$ for all $\gamma \in \Gamma$. E denotes the set of all idempotents in M. An element $a \in M$ is said to be nilpotent if $a^n = 0$ for some positive integer n. Throughout this paper by M, we mean a zero-symmetric Γ-near-ring.

The following result in [5] is given for reference:

Theorem 1. *The following are equivalent:*

(i) *M is a GGNF.*

(ii) *M is regular and each idempotent is central.*

(iii) *M is regular and sub-commutative.*

3. BASIC RESULTS

In this section we establish certain preliminary results for future use.

Proposition 1. *If M is without non-zero nilpotent elements, then M is a IFP Γ-near-ring.*

Proof. If $x\gamma y = 0$ for $x, y \in M$ and for all $\gamma \in \Gamma$, then $(y\gamma x)^2 = (y\gamma x)(y\gamma x) = y\gamma(x\gamma y)\gamma x = y\gamma 0 = 0$. This implies that $y\gamma x = 0$. Now, for $\gamma_1, \gamma_2 \in \Gamma, n \in M, (x\gamma_1 n\gamma_2 y)^2 = (x\gamma_1 n\gamma_2 y)\gamma(x\gamma_1 n\gamma_2 y) = x\gamma_1 n\gamma_2 0\gamma_1(n\gamma_2 y) = (x\gamma_1 n)\gamma_2 0 = 0$. This implies that $x\gamma_1 n\gamma_2 y = 0$. Therefore M is an IFP Γ-near-ring. □

Proposition 2. *Let M be a P(1,2) Γ-near-ring.*

(i) *If M has no non-zero nilpotent elements, then M is a right unital Γ-near-ring.*

(ii) *If M is a left unital Γ-near-ring, then M has no non-zero nilpotent elements.*

Proof. (i) Since M is a P(1,2) Γ-near-ring, we have $x\Gamma M = M\Gamma x^2$ for all $x \in M$. Now $x^2 = x\gamma x \in x\Gamma M = M\Gamma x^2$ which implies that $x^2 = m\gamma x^2$ for some $m \in M$ and for all $\gamma \in \Gamma$. This implies $(x - m\gamma x)\gamma x = 0$. Since M has no non-zero nilpotent elements and M is zero symmetric, $x\gamma(x - m\gamma x) = 0$ and $m\gamma x\gamma(x - m\gamma x) = m\gamma 0 = 0$. Now $(x - m\gamma x)^2 = (x - m\gamma x)\gamma(x - m\gamma x) = x\gamma(x - m\gamma x) - m\gamma x\gamma(x - m\gamma x) = 0$. From this and M has no non-zero nilpotent elements, we get that $x - m\gamma x = 0$ and so $x = m\gamma x \in M\Gamma x$. Thus M is a right unital Γ-near-ring.
(ii) For all $x \in M, x \in x\Gamma M = M\Gamma x^2$ for some $m \in M$ and for all $\gamma \in \Gamma$. Thus $x^2 = 0$ implies $x = 0$. Hence M has no non-zero nilpotent elements. □

Similar to the above, one can prove the following result.

Proposition 3. *Let M be a P(2,1) Γ-near-ring which is also right permutable.*

(i) *If M has no non-zero nilpotent elements, then M is a left unital Γ-near-ring.*

(ii) *If M is a right unital Γ-near-ring, then M has no non-zero nilpotent elements.*

Proposition 4. *Any homomorphic image of any P(r,m) Γ-near-ring is also a P(r,m) Γ-near-ring.*

Proof. Let M be a P(r,m) Γ-near-ring and let $f : M \to M'$ be a Γ-near-ring epimorphism. Since M is a P(r,m) Γ-near-ring, $x^r\Gamma M = M\Gamma x^m$ for all $x \in M$. Now, for $y, z \in M'$ and $\gamma \in \Gamma$, consider $y^r\gamma z = f(x)^r\gamma f(m) = f(x^r\gamma m) = f(m'\gamma' x^m) = f(m')\gamma' f(x^m) \in M'\Gamma y^m$. Therefore $y^r\Gamma M' \subseteq M'\Gamma y^m$. Similarly one can prove the other inclusion and hence $y^r\Gamma M' = M'\Gamma y^m$. □

Proposition 5. *Every left Γ subgroup of a P(1,2) Γ-near-ring is also right Γ subgroup.*

Proof. Let A be a left Γ subgroup of a P(1,2) Γ-near-ring M. For $a \in A$, $m \in M$, $\gamma \in \Gamma$, $a\gamma m \in a\Gamma M = M\Gamma a^2$ implies $a\gamma m = m'\gamma' a^2 \in M\Gamma a \subseteq M\Gamma A \subseteq A$ and so A is a right Γ-subgroup. □

The following is an immediate corollary of the above result.

Corollary 1. *Every left ideal of a P(1,2) Γ-near-ring is an ideal.*

Proposition 6. *If M is a P(1,2) or P(2,1) Γ-near-ring, then M has strong IFP.*

Proof. Let I be an ideal and $a\gamma b \in I$ for $a, b \in M$ and $\gamma \in \Gamma$. (i) Suppose M is P(1,2) Γ-near-ring. Since M is zero-symmetric, $M\Gamma I \subseteq I$. Now $a\gamma_1 m \in a\Gamma M = M\Gamma a^2$ implies $a\gamma_1 m = m'\gamma a^2$ for some $m' \in M$ and for all $\gamma \in \Gamma$. This further implies that $a\gamma_1 m\gamma_2 b = (a\gamma_1 m)\gamma_2 b = (m'\gamma a^2)\gamma_2 b = (m'\gamma a)\gamma(a\gamma_2 b) \in M\Gamma I \subseteq I$. Hence $a\gamma_1 m\gamma_2 b \in I$. Thus M has strong IFP. (ii) Let M be a P(2,1) Γ-near-ring. Consider $m\gamma_2 b \in M\Gamma b = b^2\Gamma M$. From this we get that $m\gamma_2 b = b^2\gamma m'$ for some $m' \in M$ and for all $\gamma \in \Gamma$. Now $a\gamma_1 m\gamma_2 b = a\gamma_1(m\gamma_2 b) = a\gamma_1(b^2\gamma m') = (a\gamma_1 b)\gamma(b\gamma m') \subseteq I\Gamma M \subseteq I$. Hence M has strong IFP. □

Proposition 7. *If M is a P(r,m) Γ-near-ring for some positive integers r and m, then every idempotent is central.*

Proof. Let M be a P(r,m) Γ-near-ring for some integers r and m. For $e \in E$, $e^r\Gamma M = M\Gamma e^m$ implies $e\Gamma M = M\Gamma e$. Now $e\Gamma M\Gamma e = e\Gamma(M\Gamma e) = e\Gamma M$. Hence $e\Gamma M = M\Gamma e = e\Gamma M\Gamma e$. For $m \in M$, there exists $u, v \in M$ such that $m\gamma_2 e = e\gamma_1 u\gamma_2 e$ and $e\gamma_1 m = e\gamma_1 v\gamma_2 e$. Now $e\gamma_1 m\gamma_2 e = e\gamma_1(m\gamma_2 e) = e\gamma_1(e\gamma_1 u\gamma_2 e) = e\gamma_1 u\gamma_2 e = m\gamma_2 e$ and $e\gamma_1 m\gamma_2 e = (e\gamma_1 m)\gamma_2 e = (e\gamma_1 v\gamma_2 e)\gamma_2 e = e\gamma_1 m$. Thus $e\gamma_1 m = e\gamma_1 m\gamma_2 e = m\gamma_2 e$ for all $m \in M$. Therefore every idempotent is central. □

Proposition 8. *(i) Let M be a P(1,2) Γ-near-ring. Then M is regular if and only if M is a right unital Γ-near-ring.*
(ii) Let M be a P(2,1) Γ-near-ring which is right permutable. Then M is regular if and only if M is a left unital Γ-near-ring.

Proof. (i)Assume that M is a P(1,2) Γ-near-ring and regular. For all $x \in M$, there exists $y \in M$ such that $x = x\gamma_1 y\gamma_2 x \in x\Gamma M$. Therefore M is a right unital Γ-near-ring. Conversely, let M be a right unital Γ-near-ring. For each $x \in M$, $x \in x\Gamma M = M\Gamma x^2$. From this we get that $x = m\gamma_2 x^2$ for some $m \in M$ and for all $\gamma_2 \in \Gamma$ and so $x^2 = x\gamma_1 m\gamma_2 x^2$. This further implies that $(x - x\gamma_1 m\gamma_2 x)\gamma_1 x = 0$.

From this we get that $x\gamma_1(x - x\gamma_1 m\gamma_2 x) = 0$ and $x\gamma_1 m\gamma_2 x\gamma_1(x - x\gamma_1 m\gamma_2 x) = 0$. Consider $(x - x\gamma_1 m\gamma_2 x)^2 = (x - x\gamma_1 m\gamma_2 x)\gamma_1(x - x\gamma_1 m\gamma_2 x) = x\gamma_1(x - x\gamma_1 m\gamma_2 x) - x\gamma_1 m\gamma_2 x\gamma_1(x - x\gamma_1 m\gamma_2 x) = 0$. Since M has no non-zero nilpotent elements, we get that $x - x\gamma_1 m\gamma_2 x = 0$. Hence $x = x\gamma_1 m\gamma_2 x$. i.e., M is regular. (ii) Let M be regular. Then, for each $x \in M$, there exists $y \in M$ such that $x = x\gamma_1 y\gamma_2 x \in M\Gamma x$. Therefore M is a left unital Γ-near-ring. Conversely let M be a P(2,1) left unital Γ-near-ring, which is also right permutable. Then $x \in M\Gamma x = x^2\Gamma M$ which implies that $x = x^2\gamma_2 m$ for some $m \in M$ and for all $\gamma_2 \in G$. Thus $x^2 = x^2\gamma_2 m\gamma_2 x = x\gamma_1 x\gamma_2(m\gamma_2 x) = x\gamma_1(m\gamma_2 x)\gamma_2 x$ (since M is of right permutable), which implies that $(x - x\gamma_1 m\gamma_2 x)\gamma_2 x = 0$. From this we get that $x\gamma_2(x - \gamma_1 m\gamma_2 x) = 0$ and $x\gamma_1 m\gamma_2 x\gamma_2(x - x\gamma_1 m\gamma_2 x) = 0$. Consider $(x - x\gamma_1 m\gamma_2 x)^2 = (x - x\gamma_1 m\gamma_2 x)\gamma_2(x - x\gamma_1 m\gamma - 2x) = x\gamma_2(x - x\gamma_1 m\gamma_2 x) - x\gamma_1 m\gamma_2 x\gamma_2(x - x\gamma_1 m\gamma_2 x) = 0$ and so $x = x\gamma_1 m\gamma_2 x$. Thus M is regular. □

Proposition 9. *Let M be a right unital P(1,2) Γ-near-ring. Then M is P_3 regular.*

Proof. By Proposition 8, M is regular. Thus, for $x \in M$, we have $x = x\gamma_1 m\gamma_2 x$ for some $m \in M$. Hence $x\gamma_1 m\gamma_2 x = (m\gamma_1 x^2)\gamma_1 m\gamma_2 x = (m\gamma_1 x)\gamma_2(x\gamma_1 m\gamma_2 x) = m\gamma_1 x\gamma_2 x = m\gamma_1 x^2$, which implies that $x\gamma_2(x\gamma_1 m - m\gamma_1 x) = 0$ and $x\gamma_1 m\gamma_2(x\gamma_1 m - m\gamma_1 x) = 0$ for all $m \in M$. Consider $(x\gamma_1 m - m\gamma_1 x)^2 = (x\gamma_1 m - m\gamma_1 x)\gamma_2(x\gamma_1 m - m\gamma_1 x) = x\gamma_1 m\gamma_2(x\gamma_1 m - m\gamma_1 x) - m\gamma_1 x\gamma_2(x\gamma_1 m - m\gamma_1 x) = 0$ which implies that $x\gamma_1 m = m\gamma_1 x$ for all $\gamma_1 \in \Gamma$. Hence M is P_3 regular. □

4. Generalized Gamma Near-Fields

In this section, we obtain equivalent for a Γ-near-ring to be a generalized gamma near-field.

Theorem 2. *Let M be a regular Γ-near-ring. Then the following statements are equivalent:*

(i) *M is a P(1,2) Γ-near-ring.*
(ii) *Every idempotent in M is central.*
(iii) *M is a GGNF.*
(iv) *M is a P(2,1) Γ-near-ring.*

Proof. (i) ⇒(ii) Follows from Proposition 7
(ii)⇒(iii) Follows from Theorem 3.1[5]

(iii)⇒(iv) By Theorem 3.1 [5], every idempotent is central. For $a \in M$, $a^2\gamma_1 m \in a^2\Gamma M$ for all $m \in M$. Now

$$\begin{aligned} a^2\gamma_1 m &= (a\gamma_2 a)\gamma_1 m = a^2(a\gamma_1 b\gamma_2 a)\gamma_1 m \\ &= a^2\gamma_1((b\gamma_2 a)\gamma_1 m) = a^2\gamma_1 m\gamma_1(b\gamma_2 a) \in M\Gamma a. \end{aligned}$$

Therefore $a^2\Gamma M \subseteq M\Gamma a$. For $m\gamma_1 a \in M\Gamma a$,

$$\begin{aligned} m\gamma_1 a &= m\gamma_1(a\gamma_1 b\gamma_2 a) = (a\gamma_1 b)\gamma_1 m\gamma_2 a = (a\gamma_1 b\gamma_2 a)\gamma_1(b\gamma_1 m\gamma_2 a) \\ &= a\gamma_2(a\gamma_1 b)\gamma_1(b\gamma_1 m\gamma_2 a) = a^2\gamma_1(b^2\gamma_1 m\gamma_2 a) \in a^2\Gamma M. \end{aligned}$$

Therefore $M\Gamma a = a^2\Gamma M$.
(iv) ⇒ (ii) Follows from Proposition 7.
(ii) ⇒ (i) Proof is similar to that of (iii) ⇒(iv). □

Theorem 3. *Let M be a regular Γ-near-ring. Then M is a P(r,m) Γ-near-ring for all positive integers r and m if and only if M is a P(1,2) Γ-near-ring.*

Proof. Let M be a P(1,2) Γ-near-ring. By Theorem 2, every idempotent is central. Let r and m be any two positive integers. Let $a \in x^r\Gamma M$. Then $a = x^r\gamma m$ for some $m \in M$ and for all $\gamma \in \Gamma$. Now

$$\begin{aligned} a &= x^r\gamma_2 m = (x\gamma_1 y\gamma_2 x)^r\gamma_2 m = x^r\gamma_1(y\gamma_2 x)^r\gamma_2 m \\ &= x^r\gamma_1(y\gamma_2 x)\gamma_2 m = x^r\gamma_1 m\gamma_2(y\gamma_2 x) = x^r\gamma_1 m\gamma_2(y\gamma_2 x)m \\ &= x^r\gamma_1 m\gamma_2 y m\gamma_2 x^m = (x^r\gamma_1 m\gamma_2 y m)\gamma_2 x m \in M\Gamma x^m. \end{aligned}$$

Therefore $x^r\Gamma M \subseteq M\Gamma x^m$. Similarly we can prove that $M\Gamma x^m \subseteq x^r\Gamma M$. Therefore M is a $\mathrm{P(r,m)}$ Γ-near-ring. Converse is trivial. □

Theorem 4. *If M is a left permutable as well as a right permutable regular Γ-near-ring, then it is a P(r,m) Γ-near-ring for all positive integers r and m.*

Proof. By Theorem 3, it is enough to show that M is a P(1,2) Γ-near-ring. Let M be regular. For $a \in M$, there exists $b \in M$ such that $a = a\gamma_1 b\gamma_2 a$. Now $a\gamma_1 m = (a\gamma_1 b\gamma_2 a)\gamma_1 m = (b\gamma_1 a\gamma_2 a)\gamma_1 m$. Since M is left permutable, $a\gamma_1 m = (b\gamma_1 a\gamma_2 a)\gamma_1 m = b\gamma_1 m\gamma_1 a^2$. By M is right permutable, $a\gamma_1 m \in M\Gamma a^2$. Therefore $a\Gamma M \subseteq M\Gamma a^2$. Conversely, $m\gamma_2 a^2 = m\gamma_2(a\gamma_1 b\gamma_2 a)\gamma_2 a = (m\gamma_2 a\gamma_1 b)\gamma_2 a^2 = (a\gamma_2 m\gamma_1 b)\gamma_2 a^2 = a\gamma_2(m\gamma_1 b\gamma_2 a^2 \in a\Gamma M$. Therefore $M\Gamma a^2 \subseteq a\Gamma M$. Thus $a\Gamma M = M\Gamma a^2$ for all $a \in M$. Hence M is a P(1,2) Γ-near-ring and the result follows. □

Theorem 5. *Let M be a regular P(r,m) Γ-near-ring. For two left Γ-subgroups A and B of M, the following are true.*

(i) $\sqrt{A} = A$.

(ii) $A \cap B = A\Gamma B$.

(iii) $A^2 = A\Gamma A = A$.

(iv) *If* $A \subseteq B$, *then* $A\Gamma B = A$.

(v) $A \cap M\Gamma B = A\Gamma B$.

Proof. (i) Let $x \in \sqrt{A}$. Then there exists some positive integer k such that $x^k \in A$. Since M is an right unital P(1,2) Γ-near-ring, by Theorem 4 and Proposition 6, $x \in x\Gamma M = M\Gamma x^2$ which further implies that $x = m\gamma x^2$ for some $m \in M$ and for all $\gamma \in \Gamma$. This gives that $x = m\gamma x^2 = (m\gamma x)\gamma x = (m\gamma m\gamma x^2)\gamma x = m^2\gamma x^3 = \ldots = m^{k-1}\gamma x^k \in M\Gamma A \subseteq A$. Therefore $\sqrt{A} = A$.

(ii) By Theorem 4 and Proposition 5, both A and B are right Γ-subgroups and therefore $A\Gamma B \subseteq A \cap B$. Let $x \in A \cap B, x = x\gamma_1 y\gamma_2 x \in (A\Gamma M)\Gamma B \subseteq A\Gamma B$. Thus $A \cap B = A\Gamma B$.

(iii) Taking $B = A, A = A\Gamma A = A^2$.

(iv) Follows from (ii).

(v) $A \cap M\Gamma B \subseteq A \cap B = A\Gamma B$. Therefore $A \cap (M\Gamma B) \subseteq A\Gamma B$ and $A\Gamma B \subseteq M\Gamma B$. Therefore $A\Gamma B = A \cap (M\Gamma B)$. □

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