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HYPERBOLIC CURVATURE AND K-CONVEX FUNCTIONS

TAI SUNG SONG

ABSTRACT. Let γ be a C^2 curve in the open unit disk $\mathbb{D}$. Flinn and Osgood proved that $K_{\mathbb{D}}(z,\gamma) \geq 1$ for all $z \in \gamma$ if and only if the curve $f \circ \gamma$ is convex for every convex conformal mapping f of $\mathbb{D}$, where $K_{\mathbb{D}}(z,\gamma)$ denotes the hyperbolic curvature of γ at the point z. In this paper we establish a generalization of the Flinn-Osgood characterization for a curve with the hyperbolic curvature at least 1.

1. INTRODUCTION

We begin with a brief introduction to hyperbolic regions in the complex plane $\mathbb{C}$. A general discussion of hyperbolic regions can be found in [1] and [6]. A region Ω in $\mathbb{C}$ is called hyperbolic if the complement of Ω with respect to $\mathbb{C}$ contains at least two points. Let $\mathbb{D}=\{z : |z| < 1\}$ be the open unit disk in $\mathbb{C}$. The hyperbolic metric on $\mathbb{D}$ is defined by

$$\lambda_{\mathbb{D}}(z)\,|dz| = \frac{2\,|dz|}{1-|z|^2}.$$

If a region Ω is hyperbolic, then, by the uniformization theorem [2, p.39], there is a holomorphic universal covering projection φ of $\mathbb{D}$ onto Ω. The density $\lambda_{\Omega}(z)$ of the hyperbolic metric $\lambda_{\Omega}(z)\,|dz|$ on a hyperbolic region Ω is obtained from

$$\lambda_{\Omega}(\varphi(z))\,|\varphi'(z)| = \lambda_{\mathbb{D}}(z),$$

where φ is any holomorphic universal covering projection of $\mathbb{D}$ onto Ω. The hyperbolic metric is invariant under holomorphic covering projections. In particular, the hyperbolic metric is a conformal invariant.

A hyperbolic simply connected region Ω is said to be k-convex ($k > 0$) if $|a-b| < 2/k$ for any pair of distinct points $a, b \in \Omega$ and the intersection $E_k[a,b]$ of two closed disks of radii $1/k$ that have both a and b on their boundaries lies in Ω. A hyperbolic

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simply connected region Ω is said to be 0-convex if $E_0[a,b]$ is in Ω for any pair of distinct points a and b in Ω, where $E_0[a,b]$ is the closed line segment joining a and b. We will always use convex instead of 0-convex. Mejia and Minda [4] proved that if Ω is a hyperbolic simply connected region bounded by a simple closed curve $\partial\Omega$ of class C^2 and if $K_e(z,\partial\Omega) \geq k$ for all $z \in \partial\Omega$, then Ω is k-convex. Here $K_e(z,\partial\Omega)$ denotes the euclidean curvature of $\partial\Omega$ at the point z.

Let us recall the definition of the hyperbolic curvature. For more details, see [3] and [5]. If γ is a C^2 curve in a hyperbolic region Ω with parametrization $z = z(t)$, then the hyperbolic curvature of γ at the point $z = z(t)$ is given by

$$K_\Omega(z,\gamma) = \frac{1}{\lambda_\Omega(z)}\left[K_e(z,\gamma) + 2\mathrm{Im}\left\{\frac{\partial \log \lambda_\Omega(z)}{\partial z}\frac{z'(t)}{|z'(t)|}\right\}\right],$$

where

$$K_e(z,\gamma) = \frac{1}{|z'(t)|}\mathrm{Im}\left[\frac{z''(t)}{z'(t)}\right]$$

denotes the euclidean curvature of γ at $z = z(t)$. Since $\lambda_{\mathbb{D}}(z) = 2/\left(1-|z|^2\right)$, we have

$$K_{\mathbb{D}}(z,\gamma) = \frac{1}{2}\left(1-|z|^2\right)K_e(z,\gamma) + \mathrm{Im}\left[\frac{\overline{z(t)}z'(t)}{|z'(t)|}\right]$$

for a C^2 curve $\gamma : z = z(t)$ in $\mathbb{D}$. Because the hyperbolic metric is invariant under holomorphic covering projections, so is the hyperbolic curvature. In particular, the hyperbolic curvature is conformally invariant.

Let γ be a positively oriented circle in $\mathbb{D}$ with center 0 and radius $r \in (0,1)$. A parametrization of γ is $z = z(t) = re^{it}$, $0 \leq t \leq 2\pi$. Then

$$K_e(z,\gamma) = \frac{1}{|z'(t)|}\mathrm{Im}\left\{\frac{z''(t)}{z'(t)}\right\} = \frac{1}{r}.$$

As $\overline{z(t)}z'(t)/|z'(t)| = ir$ so that

$$K_{\mathbb{D}}(z,\gamma) = \frac{1-r^2}{2}\frac{1}{r} + r = \frac{1}{2}\left(r + \frac{1}{r}\right).$$

Note that $r + \frac{1}{r} > 2$. Since the hyperbolic curvature is a conformal invariant, it follows that any circle in $\mathbb{D}$ has the hyperbolic curvature strictly larger than 1.

Let γ be the positively oriented circle $|z-a| = 1-a$ where $0 < a < 1$. This circle is internally tangent to the unit circle at the point 1. A parametrization of γ is $z = z(t) = a + (1-a)e^{it}$, $0 < t < 2\pi$. Note that $z(0) = z(2\pi) = 1 \notin \mathbb{D}$.

Then we obtain $K_e(z,\gamma) = 1/(1-a)$. Since $1-|z|^2 = 2(1-a)(a - a\cos t)$ and $\overline{z(t)}z'(t)/|z'(t)| = i(a\cos t + 1 - a)$, we obtain

$$K_{\mathbb{D}}(z,\gamma) = \frac{2(1-a)(a-a\cos t)}{2}\frac{1}{1-a} + a\cos t + 1 - a = 1.$$

Since the hyperbolic curvature is a conformal invariant, it follows that any oricycle in $\mathbb{D}$, that is, a circle internally tangent to the unit circle, has the hyperbolic curvature 1.

A conformal mapping f of the unit disk $\mathbb{D}$ is called k-convex provided $f(\mathbb{D})$ is a k-convex region. A C^2 curve γ is said to be k-convex provided $K_e(z,\gamma) \geq k$ for all $z \in \gamma$. Let γ be a C^2 curve in $\mathbb{D}$. Flinn and Osgood [3] proved that $K_{\mathbb{D}}(z,\gamma) \geq 1$ for all $z \in \gamma$ if and only if the curve $f \circ \gamma$ is convex for every convex conformal mapping f of $\mathbb{D}$. In this paper we establish a generalization of the Flinn-Osgood characterization for a curve with the hyperbolic curvature at least 1. More precisely, we prove that $K_{\mathbb{D}}(z,\gamma) \geq 1$ for all $z \in \gamma$ if and only if the curve $f \circ \gamma$ is k-convex for every k-convex conformal mapping f of $\mathbb{D}$.

2. Main Results

Let Ω be a hyperbolic region in $\mathbb{C}$. Fix $a \in \Omega$ and let $w = \varphi(z)$ be a holomorphic universal covering projection $(\mathbb{D}, 0)$ onto (Ω, a). From the identity

$$\lambda_\Omega(\varphi(z))\,|\varphi'(z)| = \frac{2}{1-|z|^2}, z \in \mathbb{D},$$

we obtain

$$\log \lambda_\Omega(\varphi(z)) + \frac{1}{2}\log \varphi'(z) + \frac{1}{2}\log \overline{\varphi'(z)} = \log 2 - \log(1 - z\overline{z}).$$

We apply the operator $\partial/\partial z$ to both sides of this identity and obtain

$$\frac{\partial \log \lambda_\Omega(\varphi(z))}{\partial w}\varphi'(z) + \frac{1}{2}\frac{\varphi''(z)}{\varphi'(z)} = \frac{\overline{z}}{1-z\overline{z}}.$$

For $z = 0$, this identity yields

(1)
$$\frac{\partial \log \lambda_\Omega(\varphi(a))}{\partial w} = -\frac{1}{2}\frac{\varphi''(0)}{\varphi'(0)^2}.$$

Mejia and Minda [4] proved that if Ω is a k-convex region, then for $z \in \Omega$

(2)
$$\left|\frac{\partial \log \lambda_\Omega(z)}{\partial z}\right| \leq \frac{1}{2}\lambda_\Omega(z)\sqrt{1 - \frac{2k}{\lambda_\Omega(z)}}$$

with equality if and only if Ω is a disk of radius $1/k$. We establish a sufficient condition for a curve in a k-convex region to be k-convex.

Theorem 1. *Let γ be a C^2 curve in a k-convex region Ω with nonvanishing tangent and $z \in \gamma$. Then $K_\Omega(z,\gamma) \geq 1$ implies $K_e(z,\gamma) > k$.*

Proof. From the definition of the hyperbolic curvature, we obtain

$$\begin{aligned} K_e(z,\gamma) &= K_\Omega(z,\gamma)\lambda_\Omega(z) - 2\,\mathrm{Im}\left\{\frac{\partial \log \lambda_\Omega(z)}{\partial z}\frac{z'(t)}{|z'(t)|}\right\} \\ &\geq K_\Omega(z,\gamma)\lambda_\Omega(z) - 2\left|\frac{\partial \log \lambda_\Omega(z)}{\partial z}\right|. \end{aligned} \tag{3}$$

Since $K_\Omega(z,\gamma) \geq 1$, it follows from (2) and (3) that

$$\begin{aligned} K_e(z,\gamma) &\geq \lambda_\Omega(z)\left(1 - \sqrt{1 - \frac{2k}{\lambda_\Omega(z)}}\right) \\ &= \frac{2k}{1 + \sqrt{1 - \frac{2k}{\lambda_\Omega(z)}}} > k. \end{aligned}$$

□

We now establish a condition for a curve in $\mathbb{D}$ to have the hyperbolic curvature at least 1.

Theorem 2. *Let γ be a C^2 curve in the open unit disk $\mathbb{D}$ with nonvanishing tangent. Then $K_\mathbb{D}(z,\gamma) \geq 1$ for all $z \in \gamma$ if and only if the curve $f \circ \gamma$ is k-convex for every k-convex conformal mapping f of $\mathbb{D}$.*

Proof. Suppose $K_\mathbb{D}(z,\gamma) \geq 1$ for all $z \in \gamma$. Let f be a k-convex conformal mapping of $\mathbb{D}$ onto a k-convex region Ω. Since the hyperbolic curvature is a conformal invariant,

$$K_\Omega(f(z), f \circ \gamma) = K_\mathbb{D}(z,\gamma) \geq 1.$$

Because Ω is k-convex, Theorem 1 yields $K_e(f(z), f\circ\gamma) \geq k$.

Conversely, suppose the curve $f \circ \gamma$ is k-convex for every k-convex conformal mapping f of $\mathbb{D}$. We note that for $\alpha > 0$, the function

$$w = f(z) = \frac{\alpha z}{1 - \sqrt{1 + \alpha k} z} = \alpha z + \alpha\sqrt{1+\alpha k} z^2 + \cdots$$

is a k-convex conformal mapping of $\mathbb{D}$. The region $\Omega = f(\mathbb{D})$ is the disk with center $-\sqrt{1+\alpha k}/k$ and radius $1/k$. Since the hyperbolic curvature is invariant under conformal mappings, we may assume that $z = 0$ without loss of generality. Furthermore, we may also assume that $-i$ is the unit tangent to γ at the origin, that is, $-i = z'(t_0)/|z'(t_0)|$, where $z(t_0) = 0$. Since $f'(0) = \alpha > 0$, it follows that $-i$ is also the unit tangent to $f \circ \gamma$ at the origin. From (1), we obtain

$$\operatorname{Im}\left\{\frac{\partial \log \lambda_\Omega(0)}{\partial w}\frac{w'(t_0)}{|w'(t_0)|}\right\} = \frac{1}{2}\frac{f''(0)}{f'(0)^2}. \tag{4}$$

Since $\lambda_\Omega(0) = 2/f'(0)$ and $K_e(0, f \circ \gamma) \geq k$, it follows from the definition of the hyperbolic curvature and (4) that

$$\begin{aligned} K_\Omega(0, f \circ \gamma) &\geq \frac{f'(0)}{2}\left[k + \frac{f''(0)}{f'(0)^2}\right] \\ &= \frac{\alpha k}{2} + \sqrt{1+\alpha k} > 1. \end{aligned}$$

Remark. If we put $k = 0$ in Theorem 2, we recover the corresponding result for convex regions which was established by Flinn and Osgood [3].

Let Δ be a disk in $\mathbb{D}$, and let γ denote the positively oriented boundary of Δ. Then γ is either a circle in $\mathbb{D}$ with $K_{\mathbb{D}}(z, \gamma) > 1$ or an oricycle in $\mathbb{D}$ with $K_{\mathbb{D}}(z, \gamma) = 1$. Thus, $K_{\mathbb{D}}(z, \gamma) \geq 1$ in all cases. Hence Theorem 2 yields the following.

Corollary 3. *Let Δ be a disk in the open unit disk $\mathbb{D}$. If f is a k-convex conformal mapping of $\mathbb{D}$, then $f(\Delta)$ is k-convex.*

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Department of Mathematics Education, Pusan National University, Pusan 609-735, Korea

Email address: `tssong@pusan.ac.kr`