J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 13, Number 3 (August 2006), Pages 227–236

NOTES ON RANDOM FIXED POINT THEOREMS

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ABSTRACT. The purpose of this paper is to establish a random fixed point theorem for nonconvex valued random multivalued operators, which generalize known results in the literature. We also derive a random coincidence fixed point theorem in the noncompact setting.

1. INTRODUCTION AND PRELIMINARIES

Random (or Stochastic) Functional Analysis began in the fifties last century with the development of functional analysis (in particular, nonlinear analysis) and probability theory. The theory of random analysis is still in the formative stage, however, research on random analysis (specifically, random equations) are mainly performed along two lines.

(I) The fundamental studies on random differential equations associated with Markov processes, initiated by Itö in 1951,

(II) The studies on classical nonlinear differential with random right-hand sides and random kernels, defined on random domains, initiated by the Prague school of Probabilist in early sixties, (see Spacek [10], Zhang [18] and references therein) who studies Fredholm integral equations with random kernels.

The study of random operator equations which forms a central topic in this discipline. The distinction between a deterministic and random approach to the formulation of operator equations lies mainly in the nature of the questions that some authors try to answer and in the interpretation of the results. The random approach permits a greater generality and flexibility than that offered by a deterministic approach. Moreover, it permits the inclusion of probabilistic feature in

Received by the editors May 24, 2006 and, in revised form, July 11, 2006.
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2000 *Mathematics Subject Classification.* 47H10.
Key words and phrases. random fixed point theorem, random coincidence fixed point theorem, Hausdorff topological vector space.

the equation, which may play an essential role in making the connection between operator equations and the real phenomena they purport to describe.

The study of random fixed points forms a central topic in this area. Random fixed point theory has received much attention since the publication of the survey article by Bhrucha-Ried [3] in 1976, in which the stochastic version of some well known theorems were proved. Since then there has been a lot of activity in this area. For detail, see [7]-[9] and [11]-[13] and the references therein.

This paper is concerned with recent research work in this field with consideration ([1], [5]-[10], [13], [17]). We establish random fixed point theorems for non-convex valued random multi-valued operator and also derive coincidence fixed point theorem in non-compact settings.

We shall use the following notation and definitions.

Let A be non-empty set. We shall denote by 2^A the family of all nonempty subsets of A. If A is a nonempty subset of a topological vector space X. We shall denote by $int_X(A)$ and $co(A)$ the interior of A in X and the convex hull of A in X respectively.

A measurable space (Ω, Σ) is a pair where Ω a set and Σ a σ-algebra of subsets of Ω. If $A \subset X$ and $\mathcal{D}$ is a nonempty family of subsets of X, we shall denote by $\mathcal{D} \cap A$ the family $\{D \cap A : D \in \mathcal{D}\}$ and by $\sigma_X(\mathcal{D})$ the smallest σ-algebra on X generated by $\mathcal{D}$. If X is a topological vector space with topology τ_X, we shall use $\mathcal{B}(X)$ to denote $\sigma_X(\tau_X)$, the Borel σ-algebra on X if there is no ambiguity on the topology τ_X.

Let $F : (\Omega, \Sigma) \to 2^X$ be a mapping. Then F is said to be measurable (resp., weakly measurable) if, for every closed (resp., open) subset B of X the set $F^{-1}(B) = \{\omega \in \Omega,\ F(\omega) \cap B \neq \emptyset\} \in \Sigma$.

Note that, if X is a metric space, the measurability implies the weak measurability. If, in addition, F is a compact-valued mapping, then the measurability is equivalent to the weak measurability.

A function $f : \Omega \to X$ is a measurable selector of F if f is a measurable and, for any $\omega \in \Omega$, $f(\omega) \in F(\omega)$. A mapping $F : \Omega \times A \to X$ is called a random operator if, for any fixed $x \in A$, the mapping $F(\cdot, x) : \Omega \to X$ is measurable. A measurable mapping $x : \Omega \to A$ is said to be a random fixed point of a random operator $F : \Omega \times A \to X$ if, for every $\omega \in \Omega$, $x(\omega) = F(\omega, x(\omega))$. A random operator $F : \Omega \times A \to X$ is said to be continuous if, for all $\omega \in \Omega$, the map $F(\omega, \cdot) : A \to X$ is continuous.

Every continuous random operator from $\Omega \times X \to Y$ is separable.

Theorem A ([6]). *Let $F : \Omega \times X \to Y$ be a separable random operator such that $F^{-1}(\omega)$ is strongly upper semicontinuous almost surely. Then $F^{-1} : \Omega \times Y \to 2^X$ is a multivalued random operator.*

Let X and Y be two topological vector spaces and $F : \Omega \times X \to 2^Y$ be a multivalued continuous random operator. The inverse of F, which is denoted by F^{-1}, is the multi-valued random operator from $\mathcal{D}(F)$, the range of F, to X defined by $x(\omega) \in F^{-1}(\omega, y(\omega))$ for any fixed $\omega \in \Omega$ if and only if

$$y(\omega) \in F(\omega, x(\omega))$$

for any fixed $\omega \in \Omega$.

Theorem B ([4, Theorem 2]). *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X and let $S : K \to 2^K$ be a multifunction such that*

(a) *for all $x \in K$, $S(x)$ is nonempty and convex,*
(b) *for all $y \in K$, $S^{-1}(y)$ is open in K.*

Then T has a fixed point, that is, there exists $x_0 \in K$ such that $x_0 \in T(x_0)$.

Theorem C ([2], [12]). *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X and $S, T : K \to 2^K$ be two multi-functions. Assume that*

(a) *for all $x \in K$, $co(S(x)) \leq T(x)$ and $S(x)$ is nonempty,*
(b) *$K = \bigcup\{int_K S^{-1}(y) : y \in K\}$.*

Then T has a fixed point, that is, there exists $x_0 \in K$ such that $x_0 \in T(x_0)$.

2. A Random Fixed Point Theorem

In this section, we prove the following random fixed point theorem.

Theorem 2.1. *Let (Ω, Σ) be a measurable space, K a nonempty convex subset of a Hausdorff topological vector space X and $S, T : \Omega \times K \to 2^K$ the two multi-valued random operator. Assume that*

(a) *for all $w \in \Omega$, such that $x(\omega) \in K$, $co(S(\omega, x(\omega))) \subseteq T(\omega, x(\omega))$ and $S(\omega, x(\omega))$ is nonempty,*
(b) *$K = \bigcup\{int_K S^{-1}(\omega, y(\omega)) : y(\omega) \in K\}$, for each fixed $\omega \in \Omega$,*

(c) *there exists a nonempty subset* B_0 *of* K *such that* B_0 *contained in a compact convex subset* B_1 *of* K *and the set* $\mathcal{D} = \bigcap\{K \backslash int_K S^{-1}(\omega, y(\omega)) : y(\omega) \in B_0$ *for any fixed* $\omega \in \Omega\}$ *is either empty or compact.*

Then there exists a measurable map $x_0 : \Omega \to K$ *such that for any* $\omega \in \Omega$, $x_0(\omega) \in T(\omega, x_0(\omega))$.

Proof. We first assume that $\mathcal{D} = \emptyset$ and define a multi-valued mapping $G : \Omega \times B_1 \to 2^{B_1}$ by $G(\omega, x(\omega)) = S(\omega, x(\omega)) \cap B_1$ for all $x(\omega) \in B_1$ and $\omega \in \Omega$.

For all $\omega \in \Omega$, $x(\omega) \in B_1$ and $G(\omega, x(\omega))$ is nonempty. Indeed, suppose that the exists $x(\omega) \in B$ such that $G(\omega, x(\omega))$ is nonempty for all $\omega \in \Omega$. Then there exists $x(\omega) \in B_1$ such that

$$S(\omega, x(\omega)) \cap B_1 = \emptyset$$

for all $\omega \in \Omega$. Hence, for all $\bar{x}(\omega) \in B_1$, $x(\omega) \notin S(x_0, x(\omega))$ and so $\bar{x}(\omega) \notin S^{-1}(\omega, x(\omega)) \supseteq int_K S^{-1}(\omega, x(\omega))$ for all $\omega \in \Omega$. This show that

$$x(\omega) \in K \backslash int_K S^{-1}(\omega, x(\omega))$$

for all $\bar{x}(\omega) \in B$. Hence $\bar{x}(\omega) \in \bigcap_{\bar{x}(\omega) \in B_1} \{K \backslash int_K S^{-1}(\omega, x(\omega))\}$ for all $\omega \in \Omega$. Therefore, $\mathcal{D}$ is nonempty, which contradicts our assumption. Moreover, we have

(a_1) for all $x(\omega) \in B_1$,

$$\begin{aligned} co(G(\omega, x(\omega))) &= co(S(\omega, x(\omega)) \cap B_1) \\ &\subseteq (co(S(\omega, x(\omega))) \cap co(B_1)) \\ &\subseteq (T(\omega, x(\omega)) \cap B_1) \\ &\subseteq T(\omega, x(\omega)) \end{aligned}$$

and hence, for any $\omega \in \Omega$, $co(G(\omega, x(\omega))) \subseteq T(\omega, x(\omega))$ for all $x(\omega) \in B_1$.

(b_1) Since $\mathcal{D} = \bigcap\{K \backslash int_K S^{-1}(\omega, y(\omega)) : \ y(\omega) \in B_0\} = \emptyset$, from the assumption (b), we

have

$$K = \bigcup\{int_K S^{-1}(\omega, y(\omega)) : \ y(\omega) \in B_0\}$$

and hence

$$K = \bigcup\{int_K S^{-1}(\omega, y(\omega)) : \ y(\omega) \in B_1\}.$$

By noting that, for any $y(\omega) \in B_1$,

$$G^{-1}(\omega, y(\omega)) = S^{-1}(\omega, y(\omega)) \cap B_1$$

and

$$int_K S^{-1}(\omega, y(\omega)) \cap B_1 \subseteq int_{B_1}(S^{-1}(\omega, y(\omega)) \cap B_1),$$

we have

$$\begin{aligned}\bigcup_{y(\omega)\in B_1} \{int_{B_1} G^{-1}(\omega, y(\omega))\} &= \bigcup_{y(\omega)\in B_1} \{int_{B_1}(S^{-1}(\omega, y(\omega)) \cap B_1)\} \\ &\supseteq \bigcup_{y(\omega)\in B_1} \{int_K(S^{-1}(\omega, y(\omega)) \cap B_1\} \\ &= K \cap B_1 \\ &= B_1\end{aligned}$$

for any $\omega \in \Omega$. Therefore, $\bigcup_{y(\omega)\in B_1} \{int_{B_1} G^{-1}(\omega, y(\omega))\} = B_1$ for any fixed $\omega \in \Omega$. Thus, from Theorem C, there exists a measurable mapping $x_0 : \omega \to B_1$ such that $x_0(\omega) \in T(\omega, x_0(\omega))$ for any $\omega \in \Omega$.

Now, we will consider the case, when $\mathcal{D}$ is a nonempty compact subset of K. Assume that random operator T has no random fixed point. We divide the remaining proof into four parts.

(1) Claim: For any fixed $\omega \in \Omega$. $K \backslash int_K S^{-1}(\omega, y(\omega)) \neq \emptyset$ for all $y(\omega) \in K$. Suppose that, for any fixed $\omega \in \Omega$, $K \backslash int_K S^{-1}(\omega, y(\omega)) = \emptyset$ for some $y(\omega) \in K$. Then we have $y(\omega) \notin K \backslash\, int_K S^{-1}(\omega, y(\omega))$, which implies that

$$y(\omega) \in int_K S^{-1}(\omega, y(\omega)) \subseteq S^{-1}(\omega, y(\omega))$$

and so

$$y(\omega) \in S(\omega, y(\omega)) \subseteq co(S(\omega, y(\omega))) \subseteq T(\omega, y(\omega))$$

for all $\omega \in \Omega$. Therefore, $y(\omega)$ is a random fixed point of random operator T, which is a contradiction of our assumption. Hence, for all $\omega \in \Omega$,

$$K \backslash\, int_K S^{-1}(\omega, y(\omega)) \neq \emptyset$$

for all $\omega \in K$.

(2) Claim: For any fixed $\omega \in \Omega$, the convex hull of each finite subset $\{y_1(\omega), y_2(\omega),$ $, y_n(\omega)\}$ of K is contained in $\bigcup_{i=1}^{n} \{K \backslash int_K S^{-1}(\omega, y_i(\omega)\}$ for all $\omega \in \Omega$. Let $\{y_1(\omega), y_2(\omega),$ $, y_n(\omega)\}$ be a finite subset of K and $\alpha_i \geq 0$ for each $i = 1, 2,$ $, n$ with $\sum_{i=1}^{n} \alpha_i = 1$. Suppose that

$$\tilde{x}(\omega) = \Sigma_{i=1}^{n} y_i(\omega) \notin \bigcup_{i=1}^{n} \{K \backslash\, int_K S^{-1}(\omega, y_i(\omega))\}$$

for all $\omega \in \Omega$. Then $\tilde{x}(\omega) \in int_K S^{-1}(\omega, y_i(\omega))$ for $i = 1, 2, \quad, n$ and any fixed $\omega \in \Omega$. Thus, for all $\omega \in \Omega$, $\tilde{x}(\omega) \in S^{-1}(\omega, y_i(\omega))$ for each $i = 1, 2, \quad, n$ and hence

$$y_i(\omega) \in S(\omega, \tilde{x}(\omega)) \subseteq co(S(\omega, \tilde{x}(\omega)))$$

for each $i = 1, 2, \quad, n$. Therefore, we have

$$\sum_{i=1}^{n} \alpha_i y_i(\omega) = \tilde{x}(\omega) \in co(S(\omega, \tilde{x}(\omega)))$$

for all $\omega \in \Omega$. This implies that, for all $\omega \in \Omega$, $\bar{x}(\omega) \in co(S(\omega, \tilde{x}(\omega))) \subseteq T(\omega, \tilde{x}(\omega))$. Thus $\tilde{x}(\omega)$ is a random fixed point of random operator T, which contradicts our assumption. Therefore, the convex hull of each finite subset $\{y_1(\omega), y_2(\omega), \quad, y_n(\omega)\}$ of K is contained in the union $\bigcup_{i=1}^{n} \{K \backslash int_K S^{-1}(\omega, y_i(\omega))\}$.

(3) Claim: For all $y(\omega) \in \Omega$,

$$\bigcap_{y(\omega) \in A} \{K \backslash\ int_K S^{-1}(\omega, y(\omega))\} \neq \emptyset,$$

where $A = co(B_1 \cup \{y_1(\omega), y_2(\omega), \quad, y_n(\omega)\})$ and $\{y_1(\omega), y_2(\omega), \quad, y_n(\omega)\}$ is a finite subset of K.

Since $A = co(B_1 \cup \{y_1(\omega), y_2(\omega), \quad, y_n(\omega)\})$ for all $\omega \in \Omega$, A is compact and convex. Suppose that, for all $\omega \in \Omega$,

$$\bigcap_{y(\omega) \in A} \{K \backslash\ int_K S^{-1}(\omega, y(\omega))\} = \emptyset.$$

Then we can define a multi-valued mapping $Q : A \times \Omega \to 2^A$ by

$$Q(\omega, x(\omega)) = \{y(\omega) \in A : x(\omega) \notin K \backslash\ int_K S^{-1}(\omega, y(\omega)) \text{ for all } \omega \in \Omega\}$$

such that $Q(\omega, x(\omega))$ is nonempty for all $x(\omega) \in A$. Then, For any fixed $\omega \in \Omega$ and $y(\omega) \in A$,

$$\begin{aligned} & Q^{-1}(\omega, y(\omega)) \\ & = \{x(\omega) \in A : \ y(\omega) \in Q(\omega, x(\omega)) \text{ for any fixed } \omega \in \Omega\} \\ & = \{x(\omega) \in A : \ x(\omega) \notin K \backslash\ int_K S^{-1}(\omega, y(\omega)) \text{ for any fixed } \omega \in \Omega\} \\ & = \{x(\omega) \in int_K S^{-1}(\omega, y(\omega)) \text{ for any fixed } \omega \in \Omega\} \\ & = int_K S^{-1}(\omega, y(\omega)) \cap A. \end{aligned}$$

We now define another multi-valued random operator $P : \Omega \times A \to 2^A$ by

$$P(\omega, x(\omega)) = co(Q(\omega, x(\omega)))$$

for all $\omega \in A$ and any fixed $\omega \in \Omega$. Now, we will show that

$$A = \bigcup_{y(\omega)\in A} \{int_A Q^{-1}(\omega, y(\omega))\}$$

for any fixed $\omega \in \Omega$. Since, for any fixed $\omega \in \Omega$,

$$\bigcap_{y(\omega)\in A} \{K\backslash\, int_K S^{-1}(\omega, y(\omega))\} = \emptyset,$$

we have

$$\bigcup_{y(\omega)\in A} \{int_K\, S^{-1}(\omega, y(\omega))\} = K$$

for any fixed $\omega \in \Omega$. Hence we have

$$\begin{aligned} A &\supseteq \bigcup_{y(\omega)\in A} \{int_A Q^{-1}(\omega, y(\omega))\} \\ &\supseteq \bigcup_{y(\omega)\in A} \{int_K S^{-1}(\omega, y(\omega)) \cap A\} \\ &= K \cap A = A. \end{aligned}$$

Therefore, by Theorem C, for any fixed $\omega \in \Omega$, there exists a measurable mapping $x_0 : \Omega \to A$, $x_0(\omega) \in A$, such that

$$x_0(\omega) \in P(\omega, x_0(\omega)) = co(Q(\omega, x_0(\omega))).$$

This implies that there exists a finite subset $\{y_1(\omega), y_2(\omega), \quad, y_n(\omega)\}$ of A such that $y_i(\omega) \in Q(\omega, x_0(\omega))$ for $i = 1, 2, \quad, k$, where $x_0(\omega) = \sum_{i=1}^{k} \alpha_i\, y_i(\omega)$, $\alpha_i \geq 0$ for $i = 1, 2, \quad, k$ and $\sum_{i=1}^{k} \alpha_i = 1$. This means that for all $\omega \in \Omega$,

$$x_0(\omega) \notin K\backslash\, int_K S^{-1}(\omega, y(\omega)) \text{ for all } i = 1, 2, \quad, k,$$

that is,

$$x_0(\omega) \in int_K S^{-1}(\omega, y(\omega)) \text{ for all } i = 1, 2, \quad, k.$$

Hence

$$x_0(\omega) = \sum_{i=1}^{k} \alpha_i y_i(\omega) \in \bigcap_{i=1}^{k} \{int_K S^{-1}(\omega, y_i(\omega))\},$$

which contradicts Claim (2). Therefore, $\bigcap_{y(\omega)\in A} \{K\backslash int_K S^{-1}(\omega, y(\omega))\} = \emptyset$ for any fixed $\omega \in \Omega$.

(4) Claim: From Claim (3), we have

$$\begin{aligned}
&\mathcal{D}\cap(\bigcap_{i=1}^{n}\{K\backslash\, int_K S^{-1}(\omega, y_i(\omega))\}) \\
&= (\bigcap_{y(\omega)\in B_0}\{K\backslash\, int_K S^{-1}(\omega, y(\omega))\})\cap(\bigcap_{i=1}^{n}\{K\backslash\, int_K S^{-1}(\omega, y_i(\omega))\}) \\
&\supseteq \bigcap_{y(\omega)\in A}\{K\backslash\, int_K S^{-1}(\omega, y(\omega))\} \text{ as } B_0\cup\{y_1(\omega), y_2(\omega), \quad , y_n(\omega)\} \\
&\subseteq A\neq\emptyset,
\end{aligned}$$

that is, for all $\omega\in\Omega$, each finite subset $\{y_1(\omega), y_2(\omega), \quad , y_n(\omega)\}$ of K, it follows that

$$\mathcal{D}\cap(\bigcap_{i=1}^{n}\{K\backslash\, int_K S^{-1}(\omega, y_i(\omega))\})\neq\emptyset.$$

Since $\mathcal{D}$ is compact and $\{K\backslash\, int_K S^{-1}(\omega, y(\omega))\}$ is closed, $\{K\backslash\, int_K S^{-1}(\omega, y(\omega))\}\cap\mathcal{D}$ is compact for all $y(\omega)\in K$ and $\omega\in\Omega$. Hence

$$\bigcap_{y(\omega)\in K}(\{K\backslash\, int_K S^{-1}(\omega, y(\omega))\}\cap D)\notin\emptyset$$

and so

$$\bigcap_{y(\omega)\in K}\{K\backslash\, int_K S^{-1}(\omega, y(\omega))\}\neq\emptyset$$

for all $\omega\in\Omega$, which contradicts the condition (b). Therefore, the random operator T has a random fixed point. □

3. A Random Coincidence Fixed Point Theorem

The following random coincidence fixed point theorem can be easily derived from Theorem 2.1.

Theorem 3.1. *Let (Ω, Σ) be a measurable space, K be a nonempty convex subset of a Hausdorff topological vector space X and $\Phi, \psi : K\times\Omega\to 2^K$ be two multi-valued random operators. Assume that the following conditions hold:*

(a) *for each $\omega\in\Omega$, $x(\omega)\in K$, $\psi^{-1}(\omega, \Phi(\omega, x(\omega)))$ is nonempty and convex,*
(b) $K=\cup\{int_K\Phi^{-1}(\omega, \psi(\omega, y(\omega))) : y(\omega)\in K$ *for all* $\omega\in\Omega\}$,
(c) *there exists a nonempty subset B_0 of K such that B_0 contained in a compact convex subset B_1 of K and the set*

$$\mathcal{D} = \cap\{K\backslash int_K \Phi^{-1}(\omega, \psi(\omega, y(\omega))) : \ y(\omega) \in B_0 \text{ for all } \omega \in \Omega\}$$

is either empty or compact.

Then there exists a measurable map $x_0 : \Omega \to K$ such that, for any $\omega \in \Omega$,

$$\Phi(\omega, x_0(\omega)) \cap \psi(\omega, x_0(\omega)) \neq \emptyset.$$

Proof. For any fixed $\omega \in \Omega$, taking $S \equiv \psi^{-1} \circ \Phi$ in Theorem 2.1 for $S \equiv T$ and $S(\omega, x(\omega))$ is convex for all $x(\omega) \in K$ and so we get the conclusion. □

Remark. In deterministic case, our problem reduces to the result of Ansari [1] and we obtain the results due to Ansari and Yao [2], Tarafdar [13], [14], Theorem 2 of Browder [4] and Tan et al. [11] as special cases.

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