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THE ASYMPTOTIC BEHAVIOR OF GENERALIZED QUADRATIC MAPPINGS ON RESTRICTED DOMAINS†

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ABSTRACT. In this paper we solve the Hyers-Ulam stability problem for quadratic functional equations on restricted domains, and then obtain an asymptotic behavior of quadratic mappings on restricted domains.

1. INTRODUCTION

In 1960 and in 1964 S. M. Ulam [17, 18] proposed the general Ulam stability problem: "When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?" The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus one can ask the following question for general functional equations: If we replace a given functional equation by a functional inequality, when can one assert that the solutions of the inequality must be close to the solutions of the given equation? If the answer is affirmative, we would say that a given functional equation is stable. In 1978 P. M. Gruber [6] remarked that Ulam's problem is of particular interest in probability theory and in the case of functional equations of different types. We wish to note that stability properties of different functional equations can have applications to unrelated fields. For instance, Zhou [19] used a stability property of the functional equation $f(x-y)+f(x+y) = 2f(x)$ to prove a conjecture

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of Z. Ditzian about the relationship between the smoothness of a mapping and the degree of its approximation by the associated Bernstein polynomials.

Now, a square norm on an inner product space satisfies the important parallelogram equality $\|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2+\|y\|^2)$ for all vectors x, y. The following functional equation which was motivated by these equations

$$Q(x+y)+Q(x-y) = 2Q(x)+2Q(y) \tag{1.1}$$

is called a quadratic functional equation, and every solution of the equation (1.1) is said to be a quadratic mapping. It is well known that a mapping Q between real vector spaces E_1, E_2 satisfies the equation (1.1) if and only if there exists a unique symmetric biadditive mapping $B : E_1 \times E_1 \to E_2$ such that $Q(x) = B(x,x)$ for all x [1]. The quadratic functional equation and several other functional equations are useful to characterize inner product spaces [2, 11, 15]. A stability theorem for the quadratic functional equation was proved by a lot of authors [4, 12, 14] and there are many interesting results concerning this problem [7, 8, 13].

In 1983 F. Skof [16] was the first author to solve the Ulam problem for additive mappings on a restricted domain. In 1998 S. Jung [9] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains. In [10], J.M. Rassias proved that if for given $r > 0$ and $\varepsilon \geq 0$ a mapping f satisfies

$$\|q(x+y)+q(x-y)-2q(x)-2q(y)\| \leq \varepsilon$$

for all $x, y \in X$ with $\|x\|+\|y\| \geq r$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\| \leq \frac{5\varepsilon}{2}$$

for all $x \in X$. Now we are going to extend the above result to more generalized equations with $(d+1)$-variables. For this purpose, we consider the following functional equations,

$$f\Big(\sum_{i=1}^{d+1} x_i\Big) + \sum_{1\leq i<j\leq d+1} f(x_i - x_j) \;=\; (d+1)\sum_{i=1}^{d+1} f(x_i), \tag{1.2}$$

$$\sum_{1\leq i<j\leq d+1} [f(x_i+x_j)+f(x_i-x_j)] \;=\; 2d\sum_{i=1}^{d+1} f(x_i) \tag{1.3}$$

for all $(d+1)$-variables $x_1, \quad , x_{d+1} \in E_1$, where $d \geq 1$ is a natural number. As a special case, these equations reduce to the equation (1.1) in the case $d = 1$. In this paper, it will be verified that the general solutions of the above functional equations are quadratic mappings in the class of functions between vector spaces. Besides we establish new theorems about the Ulam stability for the general equations and apply our results to the asymptotic behavior of functional equations on restricted domains.

2. General Solution of Eq.(1.2) and (1.3)

Let E_1 and E_2 be vector spaces. It follows that the equation (1.1) implies the equation (1.2) as follows.

Lemma 1. *A mapping $f : E_1 \to E_2$ satisfies the functional equation (1.2) or (1.3) if and only if the mapping f is quadratic.*

Proof. We first assume that f is a solution of the functional equation (1.2) or (1.3). Set $x_i := 0$ in (1.2) or (1.3) for all $i = 1, \quad , d+1$ to get $f(0) = 0$. Putting $x_i := 0$ in (1.2) or (1.3) for all $i = 3, \quad , d+1$, we get $f(x_1+x_2)+f(x_1-x_2) = 2[f(x_1)+f(x_2)]$ for all $x_1, x_2 \in E_1$.

Conversely, assume that the mapping f satisfies the functional equation (1.1). Then there exists a unique symmetric biadditive mapping $B : E_1 \times E_1 \to E_2$ such that $Q(x) = B(x, x)$ for all x. Hence it is obvious that f satisfies the equation (1.2).

On the other hand, we assume that f satisfies the equation

$$\sum_{1\le i<j\le d} [f(x_i + x_j) + f(x_i - x_j)] = 2(d-1)\sum_{i=1}^{d} f(x_i) \tag{2.1}$$

for all d-variables $x_1, \quad , x_d \in E_1$. Then we get

$$\begin{aligned}
&\sum_{1\le i<j\le d+1} [f(x_i + x_j) + f(x_i - x_j)] \\
&= \sum_{1\le i<j\le d} [f(x_i + x_j) + f(x_i - x_j)] + \sum_{i=1}^{d}[f(x_i + x_{d+1}) + f(x_i - x_{d+1})] \\
&= 2(d-1)\sum_{i=1}^{d} f(x_i) + \sum_{i=1}^{d}[2f(x_i) + 2f(x_{d+1})] \qquad (2.2) \\
&= 2d\sum_{i=1}^{d+1} f(x_i)
\end{aligned}$$

for all $(d+1)$-variables $x_1, \quad , x_{d+1} \in E_1$. Thus f satisfies the equation (1.3). This completes the proof. □

3. APPROXIMATELY QUADRATIC MAPPINGS

From now on, let X be a normed space and Y a Banach space unless we give any specific reference. Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and d a positive integer with $d \geq 1$. Now before taking up the main subject, given $f : X \to Y$, we define the difference operator $Df : X^{d+1} \to Y$ by

$$Ef(x_1, x_2, \quad , x_{d+1})$$
$$:= f\left(\sum_{i=1}^{d+1} x_i\right) + \sum_{1\leq i<j\leq d+1} f(x_i - x_j) - (d+1)\sum_{i=1}^{d+1} f(x_i)$$

for all $(d+1)$-variables $x_1, \quad , x_{d+1} \in X$, which acts as a perturbation of the equation (1.2).

Theorem 1. *Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Ef(x_1, x_2, \quad , x_{d+1})\| \leq \varepsilon(x_1, \quad , x_{d+1}) \tag{3.1}$$

for all $(d+1)$-variables $x_1, \quad , x_{d+1} \in X$, and that $\varepsilon : X^{d+1} \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{i=0}^{\infty} \frac{\varepsilon(2^i x_1, \quad , 2^i x_{d+1})}{2^{2i}}$$

converges for all $x_1, \quad , x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.2) and the inequality

$$\left\|f(x) + \frac{d^2+3d-6}{6} f(0) - Q(x)\right\| \leq \frac{1}{4}\sum_{i=0}^{\infty} \frac{\varepsilon(2^i x, 2^i x, \overbrace{0, \quad , 0}^{d-1})}{4^i} \tag{3.2}$$

for all $x \in X$. The mapping Q is defined by

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}} \tag{3.3}$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

Proof. If we take $(x_1, x_2, 0, \quad , 0)$ instead of $(x_1, \quad , x_{d+1})$ in (3.1), we obtain

$$\bigg\| f(x_1 + x_2) + f(x_1 - x_2) + \binom{d-1}{2} f(0) - 2f(x_1)$$
$$-2f(x_2) - (d+1)(d-1)f(0)\bigg\| \leq \varepsilon(x_1, x_2, 0, \quad , 0),$$

which can be rewritten in the form

$$\|q(x_1+x_2)+q(x_1-x_2)-2q(x_1)-2q(x_2)\| \le \varepsilon(x_1,x_2,0, \quad ,0), \tag{3.4}$$

for all $x_1, x_2 \in X$, where $q(x) := f(x) + \frac{(d+4)(d-1)f(0)}{4}, x \in X$.

Now applying a standard procedure of direct method (see [3, 5, 7]) to the last inequality (3.4), we see that there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.2) and the inequality

$$\left\| q(x) - \frac{q(0)}{3} - Q(x) \right\| \le \frac{1}{4} \sum_{i=0}^{\infty} \frac{\varepsilon(2^i x, 2^i x, \overbrace{0, \quad ,0}^{d-1})}{2^{2i}}$$

for all $x \in X$.

The proof of the last assertion in the theorem follows by the same reasoning as the proof of [4]. □

Theorem 2. *Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Ef(x_1, x_2, \quad , x_{d+1})\| \le \varepsilon(x_1, \quad , x_{d+1})$$

for all $(d+1)$-variables $x_1, \quad , x_{d+1} \in X$, and that $\varepsilon : X^{d+1} \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{i=1}^{\infty} 4^i \varepsilon \left(\frac{x_1}{2^i}, \quad , \frac{x_{d+1}}{2^i} \right)$$

converges for all $x_1, \quad , x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.2) and the inequality

$$\|f(x) - Q(x)\| \le \frac{1}{4} \sum_{i=1}^{\infty} 4^i \varepsilon \Big(\frac{x}{2^i}, \frac{x}{2^i}, \overbrace{0, \quad ,0}^{d-1} \Big)$$

for all $x \in X$. The mapping Q is defined by

$$Q(x) = \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

Note that one has $f(0) = 0$ in the above theorem because $\varepsilon(0, \quad ,0) = 0$ by the convergence of the series.

Given a mapping $f : X \to Y$, we define the difference operator $Df : X^{d+1} \to Y$ by

$$Df(x_1, x_2, \quad , x_{d+1}) := \sum_{1 \le i < j \le d+1} \left(\biguplus_{x_j} f(x_i) \right) - 2d \sum_{i=1}^{d+1} f(x_i)$$

for all $(d+1)$-variables $x_1, \quad , x_{d+1} \in X$, which acts as a perturbation of the equation (1.3). Furthermore, we are going to establish another theorems about the Ulam stability problem of the equation (1.3) as follows.

Theorem 3. *Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, \quad , x_{d+1})\| \le \varepsilon(x_1, \quad , x_{d+1}) \tag{3.5}$$

for all $(d+1)$-variables $x_1, \quad , x_{d+1} \in X$, and that $\varepsilon : X^{d+1} \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{i=0}^{\infty} \frac{\varepsilon(2^i x_1, \quad , 2^i x_{d+1})}{2^{2i}}$$

converges for all $x_1, \quad , x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\left\| f(x) + \frac{(d^2 + d - 3)f(0)}{3} - Q(x) \right\| \le \frac{1}{4} \sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i x)}{2^{2i}} \tag{3.6}$$

for all $x \in X$, where a mapping $\phi : X^2 \to Y$ is given by

$$\phi(x, y) := \min\left\{ \varepsilon\Big(x, 0, \quad , 0, \overbrace{y}^{i}, 0 \quad , 0\Big) \;\Big|\; 2 \le i \le d+1 \right\}. \tag{3.7}$$

The mapping Q is defined by

$$Q(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^{2n}}$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

Proof. If we take $\Big(x, 0, \quad , 0, \overbrace{y}^{i}, 0, \quad , 0\Big)$ instead of $(x_1, \quad , x_{d+1})$ in (3.5), we obtain

$$\begin{aligned}
&\|f(x+y) + f(x-y) - 2f(x) - 2f(y) - (d^2 + d - 2)f(0)\| \\
&\le \varepsilon\Big(x, 0, \quad , 0, \overbrace{y}^{i}, 0, \quad , 0\Big)
\end{aligned}$$

for all $x, y \in X$, and all i with $2 \leq i \leq d+1$, which can be written in the form

$$\|q(x+y) + q(x-y) - 2[q(x) + q(y)] - q(0)\| \leq \phi(x, y) \tag{3.8}$$

for all $x, y \in X$, where a mapping $q : X \to Y$ is defined by $q(x) := f(x) + \frac{(d^2+d-3)f(0)}{3}$ and a mapping $\phi : X^2 \to Y$ is given by (3.7). Taking $y := x$ in (3.8), we get

$$\left\| \frac{q(2x)}{4} - q(x) \right\| \leq \frac{1}{4}\phi(x, x) \tag{3.9}$$

for all $x \in X$. Now applying a standard procedure of direct method [3, 7] to the last inequality (3.9), we see that there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\|q(x) - Q(x)\| \leq \frac{1}{4} \sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i x)}{2^{2i}}$$

for all $x \in X$. □

Theorem 4. *Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, \quad, x_{d+1})\| \leq \varepsilon(x_1, \quad, x_{d+1})$$

for all $(d+1)$-variables $x_1, \quad, x_{d+1} \in X$, and that $\varepsilon : X^{d+1} \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{i=1}^{\infty} 4^i \varepsilon\left(\frac{x_1}{2^i}, \quad, \frac{x_{d+1}}{2^i}\right)$$

converges for all $x_1, \quad, x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\|f(x) - Q(x)\| \leq \frac{1}{4} \sum_{i=1}^{\infty} 4^i \phi\left(\frac{x}{2^i}, \frac{x}{2^i}\right)$$

for all $x \in X$, where a mapping $\phi : X^2 \to Y$ is given by (3.7). The mapping Q is defined by

$$Q(x) = \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

4. Approximately Quadratic Mappings on Restricted Domains

In this section we are going to investigate the Hyers-Ulam stability problem for the equations (1.2) and (1.3) on a restricted domain. As results we have an asymptotic property of the mappings concerning the equations (1.2) and (1.3).

Theorem 5. *Let $r > 0$ be fixed. Suppose that there exists a nonnegative real number ε for which a mapping $f : X \to Y$ satisfies*

$$\|Ef(x_1, x_2, \ldots, x_{d+1})\| \le \varepsilon \tag{4.1}$$

for all $(d+1)$-variables $x_1, \ldots, x_{d+1} \in X$ with $\sum_{i=1}^{d+1} \|x_i\| \ge r$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.2) and the inequality

$$\left\| f(x) + \frac{(d+4)(d-1)f(0)}{4} - Q(x) \right\| \le \frac{3\varepsilon}{2} \tag{4.2}$$

for all $x \in X$.

Proof. Taking $(x_1, \ldots, x_{d+1})$ as $(x, y, 0, \ldots, 0)$ in (4.1) with $\|x\| + \|y\| \ge r$, we obtain by the same way as (3.4)

$$\|q(x+y) + q(x-y) - 2q(x) - 2q(y)\| \le \varepsilon, \tag{4.3}$$

for all $x, y \in X$ with $\|x\| + \|y\| \ge r$, where $q(x) := f(x) + \frac{(d+4)(d-1)f(0)}{4}$. Specially, we have $\|q(0)\| \le \frac{\varepsilon}{2}$ by setting $y := 0$ and $x := t$ with $\|t\| \ge r$ in (4.3). Now, assume $\|x\| + \|y\| < r$. And choose a $t \in X$ with $\|t\| \ge 2r$. Then it holds clearly

$$\|x \pm t\| \ge r, \ \|y \pm t\| \ge r, \ \text{and } \|2t\| + \|x+y\| \ge r.$$

Therefore from (4.3) and the following functional identity

$$\begin{aligned}
&2\big[q(x+y) + q(x-y) - 2q(x) - 2q(y) - q(0)\big] \\
&= \big[q(x+y+2t) + q(x-y) - 2q(x+t) - 2q(y+t)\big] \\
&\quad + \big[q(x+y-2t) + q(x-y) - 2q(x-t) - 2q(y-t)\big] \\
&\quad + \big[-q(x+y+2t) - q(x+y-2t) + 2q(x+y) + 2q(2t)\big] \\
&\quad + \big[2q(x+t) + 2q(x-t) - 4q(x) - 4q(t)\big] \\
&\quad + \big[2q(y+t) + 2q(y-t) - 4q(y) - 4q(t)\big] \\
&\quad + \big[-2q(2t) - 2q(0) + 4q(t) + 4q(t)\big],
\end{aligned}$$

we get

$$\|q(x+y)+q(x-y)-2q(x)-2q(y)-q(0)\| \leq \frac{9\varepsilon}{2} \tag{4.4}$$

for all $x, y \in X$ with $\|x\| + \|y\| < r$. Consequently, the last functional inequality holds for all $x, y \in X$ in view of (4.3) and (4.4). Now letting $y := x$ in (4.4), we obtain

$$\|q(2x) - 4q(x)\| \leq \frac{9\varepsilon}{2}.$$

Now applying a standard procedure of direct method [5, 7] to the last inequality, we see that there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.2) and the inequality

$$\|q(x) - Q(x)\| \leq \frac{3\varepsilon}{2}$$

for all $x \in X$. □

Theorem 6. *Let $r > 0$ be fixed. Suppose that there exists a nonnegative real number ε for which a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, \quad , x_{d+1})\| \leq \varepsilon \tag{4.5}$$

for all $(d+1)$-variables $x_1, \quad , x_{d+1} \in X$ with $\sum_{i=1}^{d+1} \|x_i\| \geq r$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\left\| f(x) + \frac{(d^2+d-2)f(0)}{2} - Q(x) \right\| \leq \frac{3\varepsilon}{2} \tag{4.6}$$

for all $x \in X$.

We note that if we define

$$S_{d+1} = \{(x_1, \quad , x_{d+1}) \in X^{d+1} : \|x_i\| < r, \forall i = 1, \quad , d+1\}$$

for some fixed $r > 0$, then we have

$$\left\{ (x_1, \quad , x_{d+1}) \in X^{d+1} : \sum_{i=1}^{d+1} \|x_i\| \geq (d+1)r \right\} \subset X^{d+1} \setminus S_{d+1}.$$

Thus the following corollary is an immediate consequence of Theorem 5 and Theorem 6.

Corollary 1. *If a mapping $f : X \to Y$ satisfies the functional inequality (4.1) ((4.5), respectively) for all $(x_1, \quad , x_{d+1}) \in X^{d+1} \setminus S_{d+1}$, then there exists a unique* *quadratic mapping $Q : X \to Y$ which satisfies the equation (1.2) ((1.3), respectively)) and the inequality (4.2) ((4.6), respectively).*

From Theorem 5, we have the following corollary concerning an asymptotic property of quadratic mappings.

Corollary 2. *A mapping $f : X \to Y$ with $f(0) = 0$ is quadratic if and only if*

$$\begin{aligned} \textit{either} \quad & \|Ef(x_1, \quad , x_{d+1})\| \to 0, \\ \textit{or} \quad & \|Df(x_1, \quad , x_{d+1})\| \to 0 \end{aligned}$$

as $\sum_{i=1}^{d+1} \|x_i\| \to \infty$.

Proof. According to our asymptotic condition, there is a sequence (ε_m) decreasing to zero such that $\|Ef(x_1, \quad , x_{d+1})\| \le \varepsilon_m$ for all $(d+1)$-variables $x_1, \quad , x_{d+1} \in X$ with $\sum_{i=1}^{d+1} \|x_i\| \ge m$. Hence, it follows from Theorem 5 that there exists a unique quadratic mapping $Q_m : X \to Y$ which satisfies the equation (1.2) and the inequality

$$\|f(x) - Q_m(x)\| \le \frac{3\varepsilon_m}{2}$$

for all $x \in X$. Let m and l be positive integers with $m > l$. Then, we obtain

$$\|f(x) - Q_m(x)\| \le \frac{3\varepsilon_m}{2} \le \frac{3\varepsilon_l}{2}$$

for all $x \in X$. The uniqueness of Q_l implies that $Q_m = Q_l$ for all m, l, and so

$$\|f(x) - Q_l(x)\| \le \frac{3\varepsilon_m}{2}$$

for all $x \in X$. By letting $m \to \infty$, we conclude that f is itself quadratic.

The reverse assertion is trivial. □

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