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SEMIPRIME NEAR-RINGS WITH ORTHOGONAL DERIVATIONS

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ABSTRACT. M. Brešar and J. Vukman obtained some results concerning orthogonal derivations in semiprime rings which are related to the result that is well-known to a theorem of Posner for the product of two derivations in prime rings. In this paper, we present orthogonal generalized derivations in semiprime near-rings.

1. INTRODUCTION

A non-empty set $\mathcal{N}$ with two binary operations $+$ (addition) and $\cdot$ (multiplication) is called a *near-ring* if it satisfies the following axioms:

i) $(\mathcal{N}, +)$ is a group (not necessarily abelian),

ii) $(\mathcal{N}, \cdot)$ is a semigroup,

iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathcal{N}$.

Precisely speaking, it is a *left near-ring* because it satisfies the left distributive law. We will use the word *near-ring* to mean *left near-ring*.

An additive map $d : \mathcal{N} \to \mathcal{N}$ is called a *derivation* if the Leibniz rule

$$d(xy) = d(x)y + xd(y)$$

holds for all $x, y \in \mathcal{N}$. Furthermore, M. Brešar [2] introduced the notion of a generalized derivation. An additive map $f : \mathcal{N} \to \mathcal{N}$ is said to be a *generalized derivation* associated with a derivation d if there exists a derivation $d : \mathcal{N} \to \mathcal{N}$ such that the relation

$$f(xy) = f(x)y + xd(y)$$

holds for all $x, y \in \mathcal{N}$. Hence the concept of a generalized derivation covers both the concepts of a derivation and of a *left centralizer* (or *left multiplier*), that is, an

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additive map f satisfying $f(xy) = f(x)y$ for all $x, y \in \mathcal{N}$. This notion is found in P. Ribenboim [8], where some module structure of these higher generalized derivations was treated. Other properties of generalized derivations were given by B. Hivala [4], T. K. Lee [5] and A. Nakajima [6].

Throughout this paper, $\mathcal{N}$ will represent a zero-symmetric near-ring. Recall that $\mathcal{N}$ is *semiprime* if $x\mathcal{N}x = \{0\}$ implies $x = 0$ and is *prime* if $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$.

As for terminologies concerning near-rings used here without mention, we refer to G. Pilz [7].

Let R be a semiprime ring. Derivations d and δ in R is called orthogonal if the relation

$$d(x)R\delta(y) = 0 = \delta(y)Rd(x)$$

holds for all x, $y \in R$. M. Brešar and J. Vukman [3] introduced the notion of orthogonality for two derivations in a semiprime ring and proved some results on orthogonal derivations. N. Argaç *et. al.* [1] extended their results to orthogonal generalized derivations. In this paper, our purpose is to present orthogonal generalized derivations in semiprime *near-rings*.

2. Orthogonality of Generalized Derivations in Semiprime Near-Rings

Definition 1. Let f and g be generalized derivations of $\mathcal{N}$ associated with derivations d and δ, respectively. f and g are called *orthogonal* if the relation

$$f(x)\mathcal{N}g(y) = 0 = g(y)\mathcal{N}f(x)$$

holds for all x, $y \in \mathcal{N}$.

Example 1. Let d and δ be two derivations of $\mathcal{N}$ and $\mathcal{S} = \mathcal{N} \oplus \mathcal{N}$. Then the maps d_1 and δ_2 on the near-ring $\mathcal{S}$ which are defined by

$$d_1(x, y) = (d(x), 0) \text{ and } \delta_2(x, y) = (0, \delta(y)) \text{ for all } x, y \in \mathcal{N}$$

are derivations of $\mathcal{S}$. Assume that f and g be generalized derivations of $\mathcal{N}$ associated with derivations d and δ, respectively. Defining

$$f_1(x, y) = (f(x),\ 0) \text{ and } g_2(x, y) = (0, g(y)) \text{ for all } x, y \in \mathcal{N},$$

we see that f_1 and g_2 be generalized derivations of $\mathcal{S}$ associated with derivations d_1 and δ_2, respectively such that f_1 and g_2 are orthogonal.

We start our investigation with the following lemma corresponding to [3, Lemma 1].

Lemma 1. *Let $\mathcal{N}$ be a 2-torsion free semiprime near-ring and $a, b \in \mathcal{N}$. Then the following conditions are equivalent:*

(i) $axb = 0$ *for all* $x \in \mathcal{N}$.

(ii) $bxa = 0$ *for all* $x \in \mathcal{N}$.

(iii) $axb + bxa = 0$ *for all* $x \in \mathcal{N}$.

If one of the three conditions is fulfilled, then $ab = ba = 0$.

Proof. Suppose that $a\mathcal{N}b = 0$. Then $(b\mathcal{N}a)\mathcal{N}(b\mathcal{N}a) = 0$ as well, hence $b\mathcal{N}a = 0$ by the semiprimeness of $\mathcal{N}$. Thus (i) implies (ii). In a similar fashion we see that (ii) implies (i). Hence (ii) implies (iii).

Let us prove that (iii) implies (i). Assume that $axb + bxa = 0$ for all $x \in \mathcal{N}$. Using this identity, we obtain

$$(ayb)zayb = -(byaza)yb = ay(azb)yb = -aybzayb,$$

where $y, z \in \mathcal{N}$. Thus $2(ayb)z(ayb) = 0$ which yields $ayb = 0$ since $\mathcal{N}$ is 2-torsion-free and semiprime.

Finally, if $a\mathcal{N}b = 0$ then we also have $ab\mathcal{N}ab = 0$ and $ba\mathcal{N}ba = 0$, and therefore $ab = 0$ and $ba = 0$. The proof of the lemma is completed. □

Lemma 2. *Let f be a generalized derivation of near-ring $\mathcal{N}$ associated with a derivation d of $\mathcal{N}$. Then we have*

(i) $(f(x)y + xd(y))z = f(x)yz + xd(y)z$ *for all* $x, y, z \in \mathcal{N}$.

and

(ii) $(d(x)y + xd(y))z = d(x)yz + xd(y)z$ *for all* $x, y, z \in \mathcal{N}$.

Proof. (i). For all $x, y, z \in \mathcal{N}$, we get

$$\begin{aligned} f((xy)z) &= f(xy)z + xyd(z) \\ &= (f(x)y + xd(y))z + xyd(z). \end{aligned}$$

On the other hand,

$$\begin{aligned} f(x(yz)) &= f(x)yz + xd(yz) \\ &= f(x)yz + xd(y)z + xyd(z). \end{aligned}$$

For these two expressions of $f(xyz)$, we obtain

$$(f(x)y + xd(y))z = f(x)yz + xd(y)z$$

for all $x, y, z \in \mathcal{N}$.

(ii). It is proved by the same argument as (i). □

Lemma 3. *Let $\mathcal{N}$ be a 2-torsion free semiprime near-ring. Suppose that f (resp. g) is a generalized derivation of $\mathcal{N}$ associated with derivation d (resp. δ) of $\mathcal{N}$. If f and g are orthogonal, then the following conditions are true:*

(i) $f(x)g(y) = g(x)f(y) = 0$ *for all* $x, y \in \mathcal{N}$.

(ii) d *and* g *are orthogonal and* $d(x)g(y) = g(y)d(x) = 0$ *for all* $x, y \in \mathcal{N}$.

(iii) δ *and* f *are orthogonal and* $\delta(x)f(y) = f(y)\delta(x) = 0$ *for all* $x, y \in \mathcal{N}$.

(iv) d *and* δ *are orthogonal and this implies* $d\delta = 0$.

(v) $dg = gd = 0$ *and* $\delta f = f\delta = 0$.

(vi) $fg = gf = 0$.

Proof. (i). By the hypothesis, we have $f(x)zg(y) = 0$ for all $x, y, z \in \mathcal{N}$. Hence we get

$$f(x)g(y) = g(x)f(y) = 0$$

for all $x, y \in \mathcal{N}$ by Lemma 2.3.

(ii). Since $f(x)g(y) = 0$ and $f(x)zg(y) = 0$ for all $x, y, z \in \mathcal{N}$, we get, by using Lemma 2.4,

$$0 = f(wx)f(y) = (f(w)x + wd(x))g(y) = wd(x)g(y)$$

for all $w, x, y \in \mathcal{N}$. It follows from the semiprimeness of $\mathcal{N}$ that $d(x)g(y) = 0$ for all $x, y \in \mathcal{N}$. Then we have

$$d(xw)g(y) = (d(x)w + xd(w))g(y) = d(x)wg(y) = 0$$

for all $w, x, y \in \mathcal{N}$ in view of Lemma 2.4. Therefore by Lemma 2.3, we obtain

$$g(y)d(x) = 0$$

for all $x, y \in \mathcal{N}$ which implies (ii).

(iii). The proof is similar to (ii).

(iv). We have

$$0 = f(xz)g(yw) = (f(x)z + xd(z))(g(y)w + y\delta(w))$$

for all $w, x, y, z \in \mathcal{N}$ by (i). Thus we get $xd(z)y\delta(w) = 0$ for all $w, x, y, z \in \mathcal{N}$ by (ii) and (iii). Since $\mathcal{N}$ is semiprime, we see that $d(z)y\delta(w) = 0$ for all $w, y, z \in \mathcal{N}$, that is, d and δ are orthogonal. Then we have $d(x)y\delta(z) = 0$ for all $x, y, z \in \mathcal{N}$. Hence

$$0 = d(d(x)y\delta(z)) = d^2(x)y\delta(z) + d(x)d(y)\delta(z) + d(x)y(d\delta)(z).$$

The first two summands are zero since d and δ are orthogonal. Therefore this relation reduces to $d(x)y(d\delta)(z) = 0$ for all $x, y, z \in \mathcal{N}$. But then we also have

$$(d\delta)(z)\mathcal{N}(d\delta)(z) = 0$$

for all $z \in \mathcal{N}$. Hence $(d\delta)(z) = 0$ since $\mathcal{N}$ is semiprime, i.e., $d\delta = 0$.

(v) and (vi). Using (ii) and (iv), we obtain

$$0 = g(d(x)zg(y)) = g(d(x))zg(y) + d(x)\delta(zg(y)) = g(d(x))zg(y)$$

for all $x, y, z \in \mathcal{N}$. Replacing y by $d(x)$ in the above relation, we get $gd = 0$ by the semiprimeness of $\mathcal{N}$. Similarly, we see that since

$$d(g(x)zd(y)) = 0,\ f(\delta(x)zf(y)) = 0,\ \delta(f(x)z\delta(y)) = 0$$

and

$$g(f(x)zg(y)) = 0$$

holds for all $x, y, z \in \mathcal{N}$, we obtain $dg = f\delta = \delta f = fg = gf = 0$, respectively. Hence the proof of the theorem is completed. □

We now are ready to prove our main result.

Theorem 1. *Let $\mathcal{N}$ be a 2-torsion free semiprime near-ring. Suppose that f (resp. g) is a generalized derivation of $\mathcal{N}$ associated with derivation d(resp. δ) of $\mathcal{N}$. Then the following conditions are equivalent:*

(i) *f and g are orthogonal.*
(ii) *$f(x)g(y) = d(x)g(y) = 0$ for all $x, y \in \mathcal{N}$.*
(iii) *$f(x)g(y) = d(x)\delta(y) = 0$ for all $x, y \in \mathcal{N}$ and $dg = d\delta = 0$.*
(iv) *fg is a generalized derivation of $\mathcal{N}$ associated with a derivation $d\delta$ of $\mathcal{N}$ and $f(x)g(y) = 0$ for all $x, y \in \mathcal{N}$.*

Proof. (i) ⇒ (ii) and (iii) are proved by Lemma 2.5 (i), (ii), (iv) and (v). On the other hand, (i) ⇒ (iv) is obtained from Lemma 2.5 (i), (iv) and (vi).

(ii) ⇒ (i). Let us take xz instead of x in the relation $f(x)g(y) = 0$. Then we have from Lemma 2.4(i) that

$$0 = (f(x)z + xd(z))g(y) = f(x)zg(y)$$

for all $x, y, z \in \mathcal{N}$, hence Lemma 2.3 gives (i).

(iii) $\Rightarrow$ (i). Since $d\delta = 0$, we have

$$\begin{aligned} 0 &= (dg)(xy) \\ &= d(g(xy)) \\ &= dg(x)y + g(x)d(y) + d(x)\delta(y) + xd\delta(y) \\ &= g(x)d(y) \end{aligned}$$

for all $x, y \in \mathcal{N}$. Replacing x by xz in this relation and using Lemma 2.4(i), we get

$$0 = g(x)zd(y) + x\delta(z)d(y) = g(x)zd(y)$$

for all $x, y, z \in \mathcal{N}$. Hence we get

$$d(y)g(x) = 0$$

for all $x, y \in \mathcal{N}$ by Lemma 2.3. Then (i) follows from (ii).

(iv) $\Rightarrow$ (i). By the assumption, we have

$$(fg)(xy) = (fg)(x)y + x(d\delta)(y)$$

for all $x, y \in \mathcal{N}$ and we also obtain

$$\begin{aligned} (fg)(xy) &= f(g(xy)) \\ &= f(g(x)y + x\delta(y)) \\ &= (fg)(x)y + g(x)d(y) + f(x)\delta(y) + x(d\delta)(y) \end{aligned}$$

for all $x, y \in \mathcal{N}$. Comparing the above two results for $(fg)(xy)$, we get

$$g(x)d(y) + f(x)\delta(y) = 0 \tag{2.1}$$

for all $x, y \in \mathcal{N}$. Since $f(x)g(y) = 0$ for all $x, y \in \mathcal{N}$, we obtain

$$0 = f(x)g(yz) = f(x)g(y)z + f(x)y\delta(z) = f(x)y\delta(z)$$

for all $x, y \in \mathcal{N}$. Hence Lemma 2.3 gives $\delta(z)f(x) = 0$ for all $x, z \in \mathcal{N}$. Replacing z by yz in the last relation, we get $\delta(y)zf(x) = 0$ for all $x, y, z \in \mathcal{N}$ by Lemma 2.4(ii). Thus it follows from Lemma 2.3 that $f(x)\delta(y) = 0$ for all $x, y \in \mathcal{N}$. Now the relation (2.1) yields $g(x)d(y) = 0$ for all $x, y \in \mathcal{N}$. Putting $y = yz$ in the last relation, we have $g(x)yd(z) = 0$ for all $x, y, z \in \mathcal{N}$ which shows that $d(z)g(x) = 0$ for all $x, z \in \mathcal{N}$ by Lemma 2.3. Therefore from (ii), we obtain the result. □

Theorem 2. *Let $\mathcal{N}$ be a 2-torsion free semiprime near-ring. Suppose that f (resp. g) is a generalized derivation of $\mathcal{N}$ associated with a derivation d (resp. δ) of $\mathcal{N}$. If f and δ are orthogonal and g and d are orthogonal, then we have*

(i) $d\delta = 0$ *and* fg *is a left centralizer of* $\mathcal{N}$.

(ii) $\delta d = 0$ *and* gf *is a left centralizer of* $\mathcal{N}$.

Proof. (i). Since f and δ are orthogonal, we get $f(x)y\delta(y) = 0$ for all $x, y, z \in \mathcal{N}$. Substituting wx for x in this relation, we arrive at, by Lemma 2.4(i),

$$0 = f(wx)y\delta(z) = f(w)xy\delta(z) + wd(x)y\delta(z) = wd(x)y\delta(z)$$

for all $w, x, y, z \in \mathcal{N}$. Hence $d(x)y\delta(z) = 0$ for all $x, y, z \in \mathcal{N}$ by the semiprimeness of $\mathcal{N}$, i.e, d and δ are orthogonal. Thus we conclude from Lemma 2.5(iv) that $d\delta = 0$. Since f and δ are orthogonal and g and d are orthogonal, we get

$$f(x)\delta(y) = 0 \text{ and } g(x)d(y) = 0$$

for all $x, y \in \mathcal{N}$, respectively by Lemma 2.3. Thus we obtain

$$\begin{aligned}(fg)(xy) &= f(g(xy)) \\ &= f(g(x)y + x\delta(y)) \\ &= (fg)(x)y + g(x)d(y) + f(x)\delta(y) + x(d\delta)(y) \\ &= (fg)(x)y.\end{aligned}$$

That is, fg is a left centralizer of $\mathcal{N}$.

(ii). The proof is similar to (i). □

Theorem 3. *Let* $\mathcal{N}$ *be a 2-torsion free semiprime near-ring. Suppose that* f *is a generalized derivation of* $\mathcal{N}$ *associated with a derivation* d *of* $\mathcal{N}$. *If* $f(x)f(y) = 0$ *for all* $x, y \in \mathcal{N}$, *then we have* $f = d = 0$.

Proof. By the hypothesis, we have

$$\begin{aligned}0 &= f(x)f(yz) \\ &= f(x)(f(y)z + yd(z)) \\ &= f(x)f(y)z + f(x)yd(z) \\ &= f(x)yd(z)\end{aligned}$$

for all $x, y, z \in \mathcal{N}$. Hence we see that $d(z)f(x) = 0$ for all $x, z \in \mathcal{N}$ by Lemma 2.3. Replacing x by xz in the last relation, we get

$$0 = d(z)f(x)z + d(z)xd(z) = d(z)xd(z)$$

for all $x, z \in \mathcal{N}$. By the semiprimeness of $\mathcal{N}$, we obtain $d = 0$. Then we have from Lemma 2.4(i) and the hypothesis that

$$\begin{aligned} 0 &= f(xy)f(y) \\ &= (f(x)y + xd(y))f(y) \\ &= f(x)yf(y) + xd(y)f(y) \\ &= f(x)yf(y) \end{aligned}$$

for all $x, y \in \mathcal{N}$. Thus we get $f = 0$ by the semiprimeness of $\mathcal{N}$. □

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