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NEW IDENTITIES FOR STIRLING NUMBERS VIA RIORDAN ARRAYS

GI-SANG CHEON*, MOAWWAD E. A. EL-MIKKAWY** AND HAN-GUK SEOL***

ABSTRACT. In this paper we establish some new identities involving Stirling numbers of both kinds. These identities are obtained via Riodan arrays with a variable x. Some well-known identities are obtained as special cases of the new identities for the specific number x.

1. INTRODUCTION

We begin with some definitions and results. These results play a central role in the present work because of their convenient properties. The notations are also reproduced for those who are not familiar with it.

In terms of the *falling factorial* $(t)_n$ defined by

$$(t)_0 = 1 \quad \text{and} \quad (t)_n = \prod_{r=1}^{n}(t-r+1), n \geq 1,$$

we define the *Stirling numbers* [3]-[6], [9] of the first kind $s(n,k)$ and of the second kind $S(n,k)$ by:

$$(t)_n = \sum_{k=0}^{n} s(n,k)t^k \quad \text{and} \quad t^n = \sum_{k=0}^{n} S(n,k)(t)_k,$$

where $n \geq 0$, $s(n,0) = \delta_{n0}$, $S(n,0) = \delta_{n0}$, δ_{nk} being the Kronecker delta. It is known that [3]-[6], [9] the Stirling numbers satisfy the recurrence relations:

$$S(n,k) = S(n-1,k-1) + kS(n-1,k). \tag{1}$$

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*Corresponding author.
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If $F = (f_0, f_1, f_2, \ldots) = (f_k)_{k\geq 0}$ is a sequence of real numbers, then the function $f(t)$ defined on an interval containing zero with value at t given by the formal power series $f(t) = \sum_{k=0}^{\infty} f_k t^k$ is called the (ordinary) generating function of the sequence $(f_k)_{k\geq 0}$. In this case we write $f_k = [t^k]f(t)$, where $[t^k]$ is the coefficient operator.

A *Riordan array* [7] is an infinite lower triangular matrix defined by a couple of formal power series $D = (d_{n,k})_{n,k\geq 0} = (d(t), h(t))$, $d(0) \neq 0$ according to the rule:

$$d_{n,k} = [t^n]d(t)(th(t))^k.$$

If $h(0) \neq 0$ as well, the Riordan array is called *proper*. In this case $d_{n,n} = d(0)(h(0))^n \neq 0$ for all $n \geq 0$. Therefore any proper Riordan array has a multipicative inverse. A common example of a Riordan array is the infinite Pascal matrix P for which we have $P = \left(\frac{1}{1-t}, \frac{1}{1-t}\right)$. It is known [7] that the set of all proper Riordan arrays forms a group with the identity $I = (1, 1)$ under matrix multiplication $*$:

$$(d(t), h(t)) * (g(t), \ell(t)) = (d(t)g(th(t)), h(t)\ell(th(t))). \tag{2}$$

The following *summation property* is very useful to get new identity by using the concept of Riordan arrays:

Theorem 1.1 ([8]). *Let $D = (d(t), h(t)) = (d_{n,k})_{n,k\geq 0}$ be a Riordan array and let $f(t)$ be the generating function of a sequence $(f_k)_{k\geq 0}$. Then we have*

$$\sum_{k=0}^{n} d_{n,k} f_k = [t^n]d(t)f(th(t)). \tag{3}$$

As pointed out by Sprugnoli [8], the connection between Riordan arrays and Stirling numbers is not immediate. Therefore the main objective of the present paper is to establish new identities involving Stirling numbers via Riordan arrays. This paper is organized as follows. In section 2 the main results are given and further identities are given in section 3. A conclusion is given in section 4.

2. Riordan Arrays with a Variable and Related Identities

The $n \times n$ Pascal matrix $P_n = [\binom{n}{k}]_{n,k\geq 0}$ has been generalized by $P_n[x] = [\binom{n}{k}x^{n-k}]$ for any nonzero real number x. This generalization leads to interesting combinatorial properties and identities (see [1], [10]). For example,

$$P_n(x)P_n(y) = P_n(x+y). \tag{4}$$

Motivated by this idea, we first introduce one of the generalization of the Riordan array in a variable. Then we establish interesting identities by the summation property of the Riordan array.

Let $D = (d(t), h(t))$ be a Riordan array and x a nonzero real number. Then $D[x] := (d(xt), h(xt)) = [d_{n,k}(x)]_{n,k\geq 0}$ defines the Riordan array with a variable x such that

$$\begin{aligned} d_{n,k}(x) &= [t^n]d(xt)(th(xt))^k \\ &= [t^{n-k}]d(xt)(h(xt))^k \\ &= [t^n]d(t)(th(t))^k x^{n-k}. \end{aligned} \tag{5}$$

For example,

$$P[x] := \left(\frac{1}{1-xt}, \frac{1}{1-xt}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & \\ x & 1 & 0 & 0 & \\ x^2 & 2x & 1 & 0 & \cdots \\ x^3 & 3x^2 & 3x & 1 & \\ & & \cdots & & \end{bmatrix}. \tag{6}$$

Applying (2), we obtain

$$\left(\frac{1}{1-xt}, \frac{1}{1-xt}\right) * \left(\frac{1}{1-yt}, \frac{1}{1-yt}\right) = \left(\frac{1}{1-(x+y)t}, \frac{1}{1-(x+y)t}\right),$$

which immediately proves $P[x]P[y] = P[x+y]$ (also see (4)).

To find generalized identities, we can apply the summation property (3) to the Riordan array with variable x.

Example 1. Let $D[x] = (\frac{1}{1-xt}, \frac{1}{1-xt})$ for any nonzero real number x. If we take $f(t) = (1+t)^n$ and $f(t) = \frac{1}{(1-t)t^n}$, it is easy to obtain the following identities respectively from (3):

(i) $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 x^{n-k} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k (1-x)^{n-k},$

(ii) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} x^{n-k} = \sum_{k=0}^{2n} (-1)^k \binom{n}{k} x^k (1+x)^{2n-k} = (1+x)^n.$

The identities including the Stirling numbers of both kinds together with the binomial coefficient very frequentely arise in enumerative combinatorics, for example see [2], [9], [8], [11]. Hence it is worthwhile to investigate such identities.

Theorem 2.1. *Let $g(t)$ be the generating function of a sequence $(g_k)_{k\geq 0}$. Then we have*

$$\sum_{m=0}^{r} S(r,m)t^m g^{(m)}(t) = \sum_{k=0}^{\infty} k^r g_k t^k, r = 0,1,2,\ldots \tag{7}$$

where $S(r,m)$ is the Stirling number of the second kind and $g^{(m)}(t)$ is the m-th derivative of the function $g(t)$ with respect to t.

Proof. We prove by induction on $r \geq 1$. Let $g(t) = \sum_{k=0}^{\infty} g_k t^k$. Since

$$tg'(t) = \sum_{k=0}^{\infty} kg_k t^k,$$

(7) holds for $r = 1$. Now assume that $r \geq 2$. From (1) and $S(r,r) = S(r+1,r+1)$, we have:

$$\begin{aligned}
& t\frac{d}{dt}\sum_{m=0}^{r} S(r,m)t^m g^{(m)}(t) \\
&= t\sum_{m=0}^{r}[mS(r,m)t^{m-1}g^{(m)}(t) + S(r,m)t^m g^{(m+1)}(t)] \\
&= t[S(r,1)g'(t) + \sum_{m=2}^{r} mS(r,m)t^{m-1}g^{(m)}(t) + \sum_{m=1}^{r} S(r,m)t^m g^{(m+1)}(t)]] \\
&= tg'(t) + t\sum_{m=1}^{r-1}[S(r,m) + (m+1)S(r,m+1)]t^m g^{(m+1)}(t) + S(r,r)t^{r+1}g^{(r+1)}(t) \\
&= tg'(t) + \sum_{m=1}^{r-1} S(r+1,m+1)t^{m+1}g^{(m+1)}(t) + S(r+1,r+1)t^{r+1}g^{(r+1)}(t) \\
&= \sum_{m=0}^{r+1} S(r+1,m)t^m g^{(m)}(t),
\end{aligned}$$

which completes the proof. □

Note that Theorem 2.1 means that the function $f(t) = \sum_{m=0}^{r} S(r,m)t^m g^{(m)}(t)$ is the generating function of the sequence $(k^r g_k)_{k\geq 0}$ with the weight k^r for each $r = 0,1,\ldots$. Hence Theorem 2.1 and the summation property (3) allow us to find such new identities. Consequently we can get lots of identities for Stirling numbers whenever we take a suitable generating function $g(t)$ in (7).

Theorem 2.2. *Let x be a nonzero real number. For each $r, m = 0,1,\ldots,$ we have:*

(i) $\sum_{k=0}^{n} \binom{n}{k} k^r x^{n-k} = \sum_{k=0}^{r} k! \binom{n}{k} S(r,k)(1+x)^{n-k}$,

(ii)

$$\sum_{k=0}^{n} \binom{n}{k}\binom{m}{k} x^{n-k} k^r$$
$$= \sum_{k=0}^{r}\sum_{\ell=0}^{n-k} k! S(r,k)\binom{m}{k}\binom{m+\ell}{m}\binom{m-k}{n-k-\ell} x^{\ell}(1-x)^{n-k-\ell}.$$

Proof. Let $D[x] = (\frac{1}{1-xt}, \frac{1}{1-xt})$ and let $f(t) = \sum_{k=0}^{r} S(r,k)t^k g^{(k)}(t)$.

(i) Take $g(t) = \frac{1}{1-t}$. Then we get $d_{n,k}(x) = \binom{n}{k}x^{n-k}$ and $g_k = 1$. Hence $f_k = k^r$ where $r = 0, 1, \ldots$. Applying the summation property (3), we have:

$$\sum_{k=0}^{n} \binom{n}{k} k^r x^{n-k} = [t^n]\frac{1}{1-xt}\sum_{k=0}^{r} S(r,k)\left(\frac{t}{1-xt}\right)^k \frac{k!}{(\frac{1-xt-t}{1-xt})^{k+1}}$$
$$= \sum_{k=0}^{r} k! S(r,k)[t^n]\left(\frac{t^k}{(1-(1+x)t)^{k+1}}\right)$$
$$= \sum_{k=0}^{r} k! \binom{n}{k} S(r,k)(1+x)^{n-k},$$

which proves (i).

(ii) Take $g(t) = (1+t)^m$ for $m = 0, 1, \ldots$. Then we get $d_{n,k}(x) = \binom{n}{k}x^{n-k}$ and $g_k = \binom{m}{k}$. Hence $f_k = k^r\binom{m}{k}$ where $r = 0, 1, \ldots$. Applying the summation property (3), we have:

$$\sum_{k=0}^{n} \binom{n}{k}\binom{m}{k} x^{n-k} k^r = [t^n]\frac{1}{1-xt}\sum_{k=0}^{r} S(r,k)\left(\frac{t}{1-xt}\right)^k k!\binom{m}{k}\left(1+\frac{t}{1-xt}\right)^{m-k}$$
$$= \sum_{k=0}^{r} k! S(r,k)\binom{m}{k}[t^{n-k}]\frac{1}{(1-xt)^{m+1}}(1+(1-x)t)^{m-k}$$
$$= \sum_{k=0}^{r} k! S(r,k)\binom{m}{k}\sum_{\ell=0}^{n-k}[t^{\ell}]\frac{1}{(1-xt)^{m+1}}[t^{n-k-\ell}]$$
$$\times (1+(1-x)t)^{m-k}$$
$$= \sum_{k=0}^{r}\sum_{\ell=0}^{n-k} k! S(r,k)\binom{m}{k}\binom{m+\ell}{m}\binom{m-k}{n-k-\ell} x^{\ell}(1-x)^{n-k-\ell},$$

which proves (ii). □

From (i) and (ii) in the above theorem, we can establish lots of interesting identities involving Stirling numbers for the specific number x. Also, some well known identities are immediately reduced from it.

Corollary 2.3. *Let x be a nonzero real number. For each $r, m = 0, 1, \ldots,$ we have:*

(i) $\sum_{k=0}^{n}\binom{n}{k}k^r = \sum_{k=0}^{n}k!\binom{n}{k}S(r,k)2^{n-k} = \sum_{k=0}^{n}(n-k)!\binom{n}{k}S(r,n-k)2^k,$
$(x = 1 \textit{ in } (i))$;

(ii) $S(r,n) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^r$, $(x = -1 \textit{ in } (i))$;

(iii) $$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}x^{n-k} = \sum_{k=0}^{n}\binom{m+k}{k}\binom{m}{n-k}x^k(1-x)^{n-k}$$
$$= \sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{n}x^k(1-x)^{n-k}, \ (r = 0 \textit{ in } (ii));$$

(iv) $\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k} = \binom{m+n}{n}$, $((x,r) = (1,0) \textit{ in } (ii))$;

(v) $\sum_{k=0}^{n}k\binom{n}{k}\binom{m}{k} = n\binom{m+n-1}{n}$, $((x,r) = (1,1) \textit{ in } (ii))$;

(vi) $\sum_{k=0}^{n}k^2\binom{n}{k}\binom{m}{k} = nm\binom{m+n-2}{n-1}$, $((x,r) = (1,2) \textit{ in } (ii))$.

Note that (ii) is a third proof for Theorem 8.4, p. 289 in [2] and (iv) is the well-known Vandermonde convolution.

3. Further Identities

In this section, we establish some identities related to the Stirling numbers $s(n,k)$ of the first kind by the summation property of the Riordan concept.

(1) Let $D = (\frac{1}{1-t}, \frac{1}{1-t})$ and $f(t) = (1+t)(1+2t)\ldots(1+mt)$. In this case we obtain: $d_{n,k} = \binom{n}{k}$ and $f_k = (-1)^k s(m+1, m+1-k)$. Therefore using Theorem 2.1, we get

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}s(m+1,m+1-k) = \sum_{k=0}^{n}(-1)^k\binom{m+n-k}{m}s(m,m-k).$$

(2) Let $D = (\frac{-1}{1-t}, \frac{-1}{1-t})$ and $f(t) = (1+t)(1+2t)\ldots(1+mt)$. In this case we get $d_{n,k} = (-1)^k\binom{n}{k}$ and $f_k = (-1)^k s(m+1, m+1-k)$. Consequently applying

Theorem 2.1 gives

$$\sum_{k=0}^{n}\binom{n}{k}s(m+1,m+1-k)=\sum_{k=0}^{n}\binom{m+n-k+1}{n-k}s(m+2,m+2-k).$$

In particular, if $m=n-1$, then we have the identity:

$$\sum_{k=0}^{n}\binom{n}{k}s(n,k)=\sum_{k=0}^{n}\binom{n+k}{k}s(n+1,k+1)=\sum_{k=0}^{n}\binom{n}{k}s(n,n-k)$$

(3) Let $D=(\frac{-1}{1-t},\frac{-1}{1-t})$ and $f(t)=\frac{t^m}{(1-t)(1-2t)\dots(1-mt)}$. In this case we obtain $d_{n,k}=(-1)^k\binom{n}{k}$ and $f_k=S(k,m), k=0,1,2,\dots$. Thus using Theorem 2.1, we get

$$\sum_{k=m}^{n}(-1)^k\binom{n}{k}S(k,m)=\sum_{k=m}^{n}(-1)^kS(k-1,m-1),\ m\geq 1.$$

Substituting $m=1$ it yields the well-known identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0.$$

4. Conclusion

Although the connection between Stirling numbers and Riordan arrays is not immediate, this paper establishes new identities involving Stirling numbers of both kinds via Riordan arrays with a variable. Some well-known identities are obtained as special cases of the variable of the new identities.

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*DEPARTMENT OF MATHEMATICS, SUNGKYUNKWAN UNIVERSITY, SUWON 440-746, REPUBLIC OF KOREA
Email address: gscheon@skku.edu

**MOAWWAD E. A. EL-MIKKAWY, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, MANSOURA UNIVERSITY, MANSOURA 35516, EGYPT
Email address: mikkawy@yahoo.com

***DEPARTMENT OF MATHEMATICS, DAEJIN UNIVERSITY, POCHEON 487-711, REPUBLIC OF KOREA
Email address: hgseol@skku.edu