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ALTERNATIVE DERIVATIONS OF CERTAIN SUMMATION FORMULAS CONTIGUOUS TO DIXON'S SUMMATION THEOREM FOR A HYPERGEOMETRIC ${}_3F_2$ SERIES

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ABSTRACT. In 1994, Lavoie *et al.* have obtained twenty tree interesting results closely related to the classical Dixon's theorem on the sum of a ${}_3F_2$ by making a systematic use of some known relations among contiguous functions. We aim at showing that these results can be derived by using the same technique developed by Bailey with the help of Gauss's summation theorem and generalized Kummer's theorem obtained by Lavoie *et al.*.

Dixon gave the following classical summation formula for ${}_3F_2(1)$ (see [4, p. 92]):

$$(1)\qquad {}_3F_2\left[\begin{matrix} a, & b, & c \\ 1+a-b, & 1+a-c \end{matrix}\,\middle|\,1\right] = \frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma\left(1+a-b\right)\Gamma\left(1+a-c\right)\Gamma\left(1+\frac{a}{2}-b-c\right)}{\Gamma\left(1+a\right)\Gamma\left(1+\frac{a}{2}-b\right)\Gamma\left(1+\frac{a}{2}-c\right)\Gamma\left(1+a-b-c\right)},$$

where $\Re(a-2b-2c) > -2$.

Lavoie *et al.* [3] presented a general, artificially constructed, form of the following generalization of Dixon's theorem (1):

$$(2)\qquad f_{i,j}(a,b,c) := {}_3F_2\left[\begin{matrix} a, & b, & c \\ 1+a-b+i, & 1+a-c+i+j \end{matrix}\,\middle|\,1\right] \qquad (i=-3,\,-2,\,-1,\,0,\,1,\,2;\; j=0,\,1,\,2,\,3)$$

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by making a systematic use of the relations among contiguous functions given by Rainville [4, p. 80]. Note that $f_{0,0}(a,b,c)$ is just the Dixon's theorem (1). For example,

$$\begin{aligned} f_{0,1}(a,b,c) &:= {}_3F_2\left[\begin{matrix} a, & b, & c \\ 1+a-b, & 2+a-c \end{matrix}\,\middle|\, 1\right] \\ &= \frac{2^{-2c+1}\,\Gamma(c-1)\,\Gamma(1+a-b)\,\Gamma(2+a-c)}{\Gamma(c)\,\Gamma(a-2c+2)\,\Gamma(a-b-c+2)} \\ &\quad\cdot\left\{\frac{\Gamma\left(\frac{a}{2}-b-c+\frac{3}{2}\right)\Gamma\left(\frac{a}{2}-c+1\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a}{2}-b+\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{a}{2}-b-c+2\right)\Gamma\left(\frac{a}{2}-c+\frac{3}{2}\right)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-b+1\right)}\right\}, \end{aligned} \tag{3}$$

where $\Re(a-2b-2c) > -3$.

Here we aim at showing that 23 formulas except for the Dixon's theorem $f_{0,0}(a,b,c)$ given in (2) can be derived by using the same technique developed by Bailey [1] with the help of Gauss's theorem (see, e.g., [5, p. 45, Eq. (7)]):

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\,\middle|\, 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \qquad (\Re(c-a-b) > 0) \tag{4}$$

and generalized Kummer's theorem obtained by Lavoie *et al.* [2]:

$$\begin{aligned} &{}_2F_1\left[\begin{matrix} a, & b \\ 1+a-b+i \end{matrix}\,\middle|\, -1\right] \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1-b)\,\Gamma(1+a-b-i)}{2^a\,\Gamma\left(1-b+\frac{i}{2}+\frac{|i|}{2}\right)} \\ &\quad\cdot\left\{\frac{\mathcal{A}_i}{\Gamma\left(\frac{a}{2}-b+\frac{i}{2}+1\right)\Gamma\left(\frac{a}{2}+\frac{i}{2}+\frac{1}{2}-\left[\frac{1+i}{2}\right]\right)} + \frac{\mathcal{B}_i}{\Gamma\left(\frac{a}{2}-b+\frac{i}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{i}{2}+\frac{1}{2}-\left[\frac{i}{2}\right]\right)}\right\} \end{aligned} \tag{5}$$

for $i = 0, \pm1, \pm2, \pm3$. Here $[x]$ denotes the greatest integer less than or equal to x and $|x|$ its absolute value. The coefficients $\mathcal{A}_i$ and $\mathcal{B}_i$ are given in the following table:

TABLE 1. Table for $\mathcal{A}_i$ and $\mathcal{B}_i$

i	$\mathcal{A}_i$	$\mathcal{B}_i$
3	$3b-2a-5$	$2a-b+1$
2	$1+a-b$	-2
1	-1	1
0	1	0
-1	1	1
-2	$a-b-1$	2
-3	$2a-3b-4$	$2a-b-2$

We will prove (3) only. Then the other 23 summation formulas can be proved in a similar way. Let us start with

$$\begin{aligned}&\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}{\Gamma(1+a-b)\,\Gamma(2+a-c)}\,{}_3F_2\left[\begin{matrix}a, & b, & c\\ 1+a-b, & 2+a-c\end{matrix}\middle|\,1\right]\\ &=\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}{\Gamma(1+a-b)\,\Gamma(2+a-c)}\\ &\quad\cdot\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c+n)\,\Gamma(1+a-b)\,\Gamma(2+a-c))}{n!\,\Gamma(a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(1+a-b+n)\,\Gamma(2+a-c+n)}.\end{aligned}\tag{6}$$

For convenience, let $\mathcal{R}(a,b,c)$ be the right-hand side of (6). If we add some terms to $\mathcal{R}(a,b,c)$ and use (4), we obtain

$$\begin{aligned}\mathcal{R}(a,b,c)&=\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c+n)}{n!\,\Gamma(1+a-b+n)\,\Gamma(2+a-c+n)}\\ &\quad\cdot\frac{\Gamma(a+2n+1)\,\Gamma(a-b-c+2)}{\Gamma(a+2n+1)\,\Gamma(a-b-c+2)}\\ &=\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c+n)}{n!\,\Gamma(a+2n+1)\,\Gamma(a-b-c+2)}\,{}_2F_1\left[\begin{matrix}b+n, & c-1+n\\ & a+2n+1\end{matrix}\middle|\,1\right]\\ &=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c+n)}{n!\,\Gamma(a+2n+1)\,\Gamma(a-b-c+2)}\\ &\quad\cdot\frac{\Gamma(b+n+m)\,\Gamma(c-1+n+m)\,\Gamma(a+2n+1)}{m!\,\Gamma(b+n)\,\Gamma(c-1+n)\,\Gamma(a+2n+1+m)}.\end{aligned}$$

Then we have

$$\begin{aligned}\mathcal{R}(a,b,c)&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(c-1+n)\Gamma(a+n)\,\Gamma(b+n+m)\,\Gamma(c-1+n+m)}{n!\,m!\,\Gamma(a-b-c+2)\,\Gamma(a+2n+1+m)}\\ &=\sum_{p=0}^{\infty}\sum_{n=0}^{p}\frac{(c-1+n)\Gamma(a+n)\,\Gamma(b+p)\,\Gamma(c-1+p)}{n!\,(p-n)!\,\Gamma(a-b-c+2)\,\Gamma(a+p+1+n)}\cdot\frac{\Gamma(a)\,p!\,\Gamma(a+p+1)}{\Gamma(a)\,p!\,\Gamma(a+p+1)}.\end{aligned}$$

Separate the last double series as follows:

$$\begin{aligned}
\mathcal{R}(a,b,c) &= \sum_{p=0}^{\infty} \frac{\Gamma(b+p)\,\Gamma(c-1+p)\,\Gamma(a)}{p!\,\Gamma(a-b-c+2)\,\Gamma(a+p+1)} \\
&\quad \cdot \left\{ \sum_{n=0}^{p} \frac{(c-1)\,\Gamma(a+n)\,p!\,\Gamma(a+p+1)}{\Gamma(a)\,n!\,(p-n)!\,\Gamma(a+p+1+n)} \right. \\
&\qquad \left. + \sum_{n=0}^{p} \frac{n\,\Gamma(a+n)\,p!\,\Gamma(a+p+1)}{\Gamma(a)\,n!\,(p-n)!\,\Gamma(a+p+1+n)} \right\} \\
&= \sum_{p=0}^{\infty} \frac{\Gamma(b+p)\,\Gamma(c-1+p)\,\Gamma(a)}{p!\,\Gamma(a-b-c+2)\,\Gamma(a+p+1)} \\
&\quad \cdot \left\{ \sum_{n=0}^{p} \frac{(c-1)\,\Gamma(a+n)\,p!\,\Gamma(a+p+1)}{\Gamma(a)\,n!\,(p-n)!\,\Gamma(a+p+1+n)} \right. \\
&\qquad \left. + \sum_{n=1}^{p} \frac{\Gamma(a+n)\,p!\,\Gamma(a+p+1)}{\Gamma(a)\,(n-1)!\,(p-n)!\,\Gamma(a+p+1+n)} \right\}.
\end{aligned}$$

Using the Pochammer symbol

$$(z)_n := \frac{\Gamma(z+n)}{\Gamma(z)} \quad (n \in \mathbb{N}_0) \quad \text{and} \quad (p-n)! = \frac{(-1)^n\,p!}{(-p)_n} \quad (0 \le n \le p),$$

we get

$$\begin{aligned}
\mathcal{R}(a,b,c) &= \sum_{p=0}^{\infty} \frac{\Gamma(b+p)\,\Gamma(c-1+p)\,\Gamma(a)}{p!\,\Gamma(a-b-c+2)\,\Gamma(a+p+1)} \left\{ (c-1) \sum_{n=0}^{p} \frac{(a)_n\,(-p)_n\,(-1)^n}{(a+p+1)_n\,n!} \right. \\
&\quad \left. + \frac{ap}{a+p+1} \sum_{n=0}^{p-1} \frac{(a+1)_n\,(1-p)_n\,(-1)^n}{(a+p+2)_n\,n!} \right\} \\
&= \sum_{p=0}^{\infty} \frac{\Gamma(b+p)\,\Gamma(c-1+p)\,\Gamma(a)}{p!\,\Gamma(a-b-c+2)\,\Gamma(a+p+1)} \left\{ (c-1)\,{}_2F_1\left[\begin{matrix} a, & -p \\ \multicolumn{2}{c}{1+a+p} \end{matrix} \,\middle|\, -1\right] \right. \\
&\quad \left. + \frac{ap}{a+p+1}\,{}_2F_1\left[\begin{matrix} a+1, & 1-p \\ \multicolumn{2}{c}{2+a+p} \end{matrix} \,\middle|\, -1\right] \right\}.
\end{aligned}$$

Applying the cases $i=0$ and $i=1$ of (5) to the ${}_2F_1(-1)$'s, after a simplification, we obtain

$$\mathcal{R}(a,b,c) = \frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c-1)\,\Gamma\left(\frac{1}{2}\right)}{2^a\,\Gamma(a-b-c+2)}$$

$$\cdot\left\{\frac{c-1-\frac{a}{2}}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+1\right)}\,{}_2F_1\left[\begin{matrix}b, & c-1\\ & \frac{a}{2}+1\end{matrix}\,\middle|\,1\right]\right.$$
$$\left.+\frac{a}{2\,\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+1\right)}\,{}_2F_1\left[\begin{matrix}b, & c-1\\ & \frac{a}{2}+\frac{1}{2}\end{matrix}\,\middle|\,1\right]\right\}.$$

Now apply the Gauss's theorem (4) to ${}_2F_1(1)$'s in the last result obtained. Finally equating (6) and the last result proves (3).

We conclude this paper by remarking that $\{f_{i,1}(a,b,c)\,|\,i = -3,\ -2,\ -1,\ 1,\ 2\}$ can be proved in the exactly same method as in the above proof of $f_{0,1}(a,b,c)$, and the other classes $\{f_{i,0}(a,b,c)\,|\,i = -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3\}$, $\{f_{i,2}(a,b,c)\,|\,i = -3,\ -2,\ -1,\ 0,\ 1,\ 2\}$, and $\{f_{i,3}(a,b,c)\,|\,i = -3,\ -2,\ -1,\ 0\}$ can also be proved in the exactly same method among their respective families, yet in a similar way as in the above illustration.

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