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ON THE STABILITY OF THE COSINE TYPE FUNCTIONAL EQUATION

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ABSTRACT. The aim of this paper is to study the superstability problem of the cosine type functional equation $f(x+y)+f(x+\sigma y)=2g(x)g(y)$.

1. INTRODUCTION

J. Baker, J. Lawrence and F. Zorzitto in [4] introduced that if f satisfies the stability inequality $|E_1(f)-E_2(f)|\le\varepsilon$, then either f is bounded or $E_1(f)=E_2(f)$. This is now frequently referred to as *superstability.*

The superstability of the cosine functional equation (also called the d'Alembert functional equation)

$$f(x+y)+f(x-y)=2f(x)f(y) \tag{C}$$

and the sine functional equation

$$f(x)f(y)=f\Big(\frac{x+y}{2}\Big)^2-f\Big(\frac{x-y}{2}\Big)^2, \tag{S}$$

are investigated by J. Baker [3] and P. W. Cholewa [6], respectively.

The d'Alembert functional equation (C) is generalized to the following functional equations

$$f(x+y)+f(x-y)=2f(x)g(y), \tag{C_{fg}}$$

$$f(x+y)+f(x-y)=2g(x)f(y), \tag{C_{gf}}$$

which f,g are two unknown functions to be determined. The equation (C_{fg}), raised by Wilson, is called the Wilson equation, and the equation (C_{gf}) is raised by Pl.

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Kannappan and G. H. Kim [9]. The above functional equations have been investigated by Badora, Ger, Kannappan, Kim, Sinopoulos, Stetkaer, etc ([2], [3], [6], [7], [8], [10], [11], [12], [13], [14], [15]).

As another generalization of the cosine functional equation, we can consider the cosine type functional equations as follows :

$$(C_{gg}) \qquad f(x+y)+f(x-y)=2g(x)g(y),$$

$$(\widetilde{C}_{gg}) \qquad f(x+y)+f(x+\sigma y)=2g(x)g(y).$$

Given mappings $f,g:G\to\mathbb{C}$, we define a difference $D\widetilde{C}_{gg}(x,y):G\times G\to\mathbb{C}$ as

$$D\widetilde{C}_{gg}(x,y):=f(x+y)+f(x+\sigma y)-2g(x)g(y),$$

and let

$$(\widetilde{C}) \qquad f(x+y)+f(x+\sigma y)=2f(x)f(y),$$

$$(\widetilde{S}) \qquad f(x)f(y)=f\Big(\frac{x+y}{2}\Big)^2-f\Big(\frac{x+\sigma y}{2}\Big)^2.$$

$$(\widetilde{C}_{fg}) \qquad f(x+y)+f(x+\sigma y)=2f(x)g(y),$$

$$(\widetilde{C}_{gf}) \qquad f(x+y)+f(x+\sigma y)=2g(x)f(y).$$

Badora and Ger [2], Kim and Dragomir [11], Kim [10] have proved the superstability of the cosine equation (C), the cosine type equations (C_{fg}), (C_{gf}), $(\widetilde{C}_{fg})$, and $(\widetilde{C}_{gf})$ under the condition

$$|DC(x,y)|\le\varphi(x) \quad \text{or} \quad \varphi(y),$$

$$|DC_{fg}(x,y)|\le\varphi(x) \quad \text{or} \quad \varphi(y),$$

$$|DC_{gf}(x,y)|\le\varphi(x) \quad \text{or} \quad \varphi(y),$$

$$|D\widetilde{C}_{fg}(x,y)|\le\varphi(x) \quad \text{or} \quad \varphi(y),$$

$$|D\widetilde{C}_{gf}(x,y)|\le\varphi(x) \quad \text{or} \quad \varphi(y).$$

The aim of this paper is to investigate the stability problem on the abelian group for the cosine type functional equation $(\widetilde{C}_{gg})$ under the condition

$$|D\widetilde{C}_{gg}(x,y)|\le\varphi(x) \quad \text{or} \quad \varphi(y).$$

The obtained results not only imply naturally the stability of the equations (C), $(\widetilde{C})$ and (C_{gg}), but also can be extended their range to the Banach algebra.

In this paper, let $(G,+)$ be an Abelian group, $\mathbb{C}$ the field of complex numbers, and $\mathbb{R}$ the field of real numbers, and let σ be an endomorphism of G with $\sigma(\sigma(x))=x$ for all $x\in G$, where a notation $\sigma(x)=\sigma x$. The properties $g(x)=g(\sigma x)$ and

$g(x) = -g(\sigma x)$ with respect to σ will be represented as even and odd functions for convenience, respectively. We may assume that f and g are nonzero functions and ε is a nonnegative real constant, $\varphi : G \to \mathbb{R}$. If, in all the results of this article, we consider the Kannappan condition $f(x+y+z) = f(x+z+y)$(see [8]), then we will obtain same results for the semigroup $(G, +)$.

2. Stability of the Equation $(\widetilde{C}_{gg})$

In this section, we will investigate the stability of the cosine type functional equation $(\widetilde{C}_{gg})$ related to the sine functional equation $(\widetilde{C})$.

Theorem 1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

(1) $$|f(x+y) + f(x+\sigma y) - 2g(x)g(y)| \le \varphi(y) \qquad x, y \in G,$$

with $g(0) = 1$. Then either g is bounded or g satisfies $(\widetilde{C})$.

Proof. Let g be unbounded. Then we can choose a sequence $\{x_n\}$ in G such that

(2) $$0 \neq |g(x_n)| \to \infty, \quad \text{as} \quad n \to \infty$$

Taking $x = x_n$ in (1) we obtain

$$\left| \frac{f(x_n + y) + f(x_n + \sigma y)}{2g(x_n)} - g(y) \right| \le \frac{\varphi(y)}{|2g(x_n)|},$$

that is,

(3) $$\lim_{n \to \infty} \frac{f(x_n + y) + f(x_n + \sigma y)}{2g(x_n)} = g(y) \quad y \in G.$$

Using (1) we have

$$\begin{aligned} &|f((x_n + x) + y) + f((x_n + x) + \sigma y) - 2g(x_n + x)g(y) \\ &\quad + f((x_n + \sigma x) + y) + f((x_n + \sigma x) + \sigma y) - 2g(x_n + \sigma x)g(y)| \\ &\le 2\varphi(y) \end{aligned}$$

so that

$$\begin{aligned} &\left| \frac{f(x_n + (x+y)) + f(x_n + \sigma(x+y))}{2g(x_n)} \right. \\ &\quad \left. + \frac{f(x_n + (x+\sigma y)) + f(x_n + \sigma(x+\sigma y))}{2g(x_n)} - 2\frac{g(x_n + x) + g(x_n + \sigma x)}{2g(x_n)} g(y) \right| \\ &\le \frac{\varphi(y)}{|g(x_n)|} \end{aligned}$$

for all $x, y \in G$. Taking the limit as $n \longrightarrow \infty$ with the use of (2) and (3), we conclude that, for every $x \in G$, there exists the limit

$$h(x) := \lim_{n\to\infty} \frac{g(x_n + x) + g(x_n + \sigma x)}{2g(x_n)},$$

where the function $h : G \to \mathbb{C}$ obtained in that way has to satisfy the equation

$$g(x+y) + g(x+\sigma y) = 2h(x)g(y) \quad x, y \in G. \tag{4}$$

The assumption $g(0) = 1$ forces, which jointly with (4), that $g = h$. Namely this means that g is solution of $(\widetilde{C})$. Therefore the proof of the theorem is completed. □

Corollary 1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + f(x+\sigma y) - 2g(x)g(y)| \le \varepsilon \qquad x, y \in G,$$

with $g(0) = 1$. Then either g is bounded or g satisfies $(\widetilde{C})$.

Corollary 2. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + f(x-y) - 2g(x)g(y)| \le \varphi(y) \qquad x, y \in G,$$

with $g(0) = 1$. Then either g is bounded or g satisfies (C).

Corollary 3. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + f(x-y) - 2g(x)g(y)| \le \varepsilon \qquad x, y \in G,$$

with $g(0) = 1$. Then either g is bounded or g satisfies (C).

Theorem 2. *Let $(G, +)$ be an uniquely 2-divisible group. Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + f(x+\sigma y) - 2g(x)g(y)| \le \varphi(x) \qquad x, y \in G, \tag{5}$$

with $g(0) = 0$. Then either g is bounded or g satisfies $(\widetilde{S})$.

Proof. Let g be unbounded. Then we can choose a sequence $\{y_n\}$ in G such that

$$0 \ne |g(y_n)| \to \infty, \quad \text{as} \quad n \to \infty \tag{6}$$

Taking $y = y_n$ in (5) we obtain

$$\left| \frac{f(x+y_n) + f(x+\sigma y_n)}{2g(y_n)} - g(x) \right| \le \frac{\varphi(x)}{|2g(y_n)|},$$

that is,

$$\lim_{n\to\infty} \frac{f(x+y_n) + f(x+\sigma y_n)}{2g(y_n)} = g(x) \quad y \in G. \tag{7}$$

Using (5) we have

$$\begin{aligned}&|f(x+(y+y_n))+f(x+\sigma(y+y_n))-2g(x)g(y+y_n)\\&\quad+f(x+(\sigma y+y_n))+f(x+\sigma(\sigma y+y_n))-2g(x)g(\sigma y+y_n)|\\&\leq 2\varphi(x)\end{aligned}$$

so that

$$\begin{aligned}&\Bigg|\frac{f(x+y+y_n)+f(x+y+\sigma y_n)}{2g(y_n)}\\&\quad+\frac{f(x+\sigma y+y_n)+f(x+\sigma y+\sigma y_n)}{2g(y_n)}-2g(x)\frac{g(y+y_n)+g(\sigma y+y_n)}{2g(y_n)}\Bigg|\\&\leq\frac{\varphi(x)}{|g(y_n)|}\end{aligned}$$

for all $x, y \in G$. Taking the limit as $n \longrightarrow \infty$ with the use of (6) and (7), we conclude that, for every $x \in G$, there exists the limit

$$h(y) := \lim_{n\to\infty}\frac{g(y+y_n)+g(\sigma y+y_n)}{2g(y_n)},$$

where the function $h : G \to \mathbb{C}$ satisfies the equation

$$g(x+y)+g(x+\sigma y)=2g(x)h(y) \quad x, y \in G. \tag{8}$$

The assumption $g(0) = 0$ forces, which jointly with (8), that g is odd function. Keeping this in mind, by means of (8), we infer the equality

$$\begin{aligned}g(x+y)^2-g(x+\sigma y)^2&=[g(x+y)+g(x+\sigma y)][g(x+y)-g(x+\sigma y)]\\&=2g(x)[g(x+y)-g(x+\sigma y)]h(y)\\&=g(x)\big[g(x+2y)-g(x+2\sigma y)\big]\\&=g(x)\big[g(x+2y)+g(\sigma x+2y)\big]\\&=g(x)2g(2y)h(x).\end{aligned}$$

Since the oddness of g forces $g(x+\sigma x) = 0$, putting $x = y$ in (8) we get the equation

$$g(2y) = 2h(y)g(y), \quad \forall y \in G.$$

This, in return, leads to the equation

$$g(x+y)^2-g(x+\sigma y)^2=g(2x)g(2y)$$

valid for all $x, y \in G$ which, in the light of the unique 2-divisibility of G, states nothing else but $(\widetilde{S})$. Therefore the proof of the theorem is completed. □

Corollary 4. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + f(x+\sigma y) - 2g(x)g(y)| \le \varepsilon \qquad x, y \in G,$$

with $g(0) = 0$. Then either g is bounded or g satisfies $(\widetilde{S})$.

Corollary 5. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + f(x-y) - 2g(x)g(y)| \le \varphi(x) \qquad x, y \in G,$$

with $g(0) = 0$. Then either g is bounded or g satisfies $(\widetilde{S})$.

3. Extension to the Banach Algebra

In all results of section 2, the range of function on the abelian group can be extended to the Banach algebra. To simplify, we will combine two theorems in one.

Theorem 3. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \to E$ satisfy one of the inequalities*

$$\|f(x+y) + f(x+\sigma y) - 2g(x)g(y)\| \le \begin{cases} (i)\ \varphi(y) \\ (ii)\ \varphi(x) \end{cases} \qquad x, y \in G, \tag{9}$$

with $g(0) = 1$ in the case (i) and $g(0) = 0$ in the case (ii).

For an arbitrary linear multiplicative functional $x^* \in E^*$,

if the superposition $x^* \circ g$ fails to be bounded, then

(a) *g satisfies $(\widetilde{C})$ in the case (i),*

(b) *g satisfies $(\widetilde{S})$ under the 2-divisible group G in the case (ii).*

Proof. Assume that (9) holds and fix arbitrarily a linear multiplicative functional $x^* \in E^*$. Since $\|x^*\| = 1$, we have, for every $x, y \in G$

$$\begin{aligned} \varphi(x)(\text{ or } \varphi(y)) &\ge \|f(x+y) + f(x+\sigma y) - 2g(x)g(y)\| \\ &= \sup_{\|y^*\|=1} \left| y^*\big(f(x+y) + f(x+\sigma y) - 2g(x)g(y)\big) \right| \\ &\ge \left| x^*\big(f(x+y)\big) + x^*\big(f(x+\sigma y)\big) - 2x^*\big(g(x)\big)x^*\big(g(y)\big) \right|, \end{aligned}$$

which states that the superposition $x^* \circ g$ yields a solution of the inequality (1) and the inequality (5). Since, by assumption, the superposition $x^* \circ g$ is unbounded, an appeal to Theorem 1 and Theorem 2 shows that the function $x^* \circ g$ solves the equation $(\widetilde{C})$ in (i) and the function $x^* \circ g$ solves the equation $(\widetilde{S})$ in (ii), respectively.

In other words, bearing the linear multiplicativity of x^* in mind, for all $x, y \in G$, the differences $\widetilde{C}(x,y)$ and $\widetilde{S}(x,y)$ fall into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$\widetilde{C}(x,y), \widetilde{S}(x,y) \in \bigcap \{\ker x^* : x^* \text{ is a multiplicative member of } E^*\}$$

for all $x, y \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e.

$$\widetilde{C}(x,y) = \widetilde{S}(x,y) = 0 \quad \text{for all} \quad x, y \in G,$$

as claimed. □

Corollary 6. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \to E$ satisfy one of the inequalities*

$$\|f(x+y) + f(x-y) - 2g(x)g(y)\| \leq \begin{cases} (i)\varphi(y) \\ (ii)\varphi(x) \end{cases} \qquad x, y \in G,$$

with $g(0) = 1$ in the case (i) and $g(0) = 0$ in the case (ii).

For an arbitrary linear multiplicative functional $x^* \in E^*$,
if the superposition $x^* \circ g$ fails to be bounded, then
(a) *g satisfies (C) in the case (i),*
(b) *g satisfies (S) under the 2-divisible group G in the case (ii).*

Corollary 7. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \to E$ satisfy the inequality*

$$\|f(x+y) + f(x+\sigma y) - 2g(x)g(y)\| \leq \varepsilon \qquad x, y \in G.$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,
if the superposition $x^* \circ g$ fails to be bounded, then
(a) *g satisfies $(\widetilde{C})$ under the condition $g(0) = 1$,*
(b) *g satisfies $(\widetilde{S})$ under the 2-divisible group G and the condition $g(0) = 0$.*

Corollary 8. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \to E$ satisfy the inequality*

$$\|f(x+y) + f(x-y) - 2g(x)g(y)\| \leq \varepsilon \qquad x, y \in G.$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,
if the superposition $x^* \circ g$ fails to be bounded, then
(a) *g satisfies (C) under the condition $g(0) = 1$,*
(b) *g satisfies (S) under the 2-divisible group G and the condition $g(0) = 0$.*

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