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A NOTE ON THE ROOT SPACES OF AFFINE LIE ALGEBRAS OF TYPE $D_l^{(1)}$

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ABSTRACT. Let $\mathfrak{g} = \mathfrak{g}(A) = \mathfrak{N}_- \oplus \mathfrak{h} \oplus \mathfrak{N}_+$ be a symmetrizable Kac-Moody Lie algebra of type $D_l^{(1)}$ with W as its Weyl group. We construct a sequence of root spaces with certain conditions. We also find the number of terms of this sequence is less then or equal to the hight of θ, the highest root.

0. INTRODUCTION

We first recall some of the basic definitions of Kac-Moody Lie algebras.

Let $A = (a_{ij})_{i,j \in I}$ be an indecomposible generalized Cartan matrix and $\mathfrak{g} = \mathfrak{g}(A)$ denote the associated Kac-Moody Lie algebra over the field of complex numbers. Following the usual convention, we will take the index set I to be $\{0, 1, \ldots, l\}$ when A is of affine type and I to be $\{1, 2, \ldots, l\}$ otherwise. Let $\mathfrak{g} = \mathfrak{N}_- \oplus \mathfrak{h} \oplus \mathfrak{N}_+$ be its triangular decomposition with respect to the Cartan subalgebra $\mathfrak{h}$ and let $\Delta = \Delta_+ \cup \Delta_-$ denote the set of roots with Δ_+ and Δ_- denoting the set of positive and negative roots respectively. Let $\Pi = \{\alpha_i | i \in I\}$ denote the set of simple roots, $\check{\Pi} = \{\check{\alpha}_i | i \in I\}$ denote the set of simple coroots. Let $\mathfrak{g}_\alpha$ denote the α-root space and $Q = \sum_{i=0}^{n} Z\alpha_i$ denote the root lattice. Let W be the Weyl group of $\mathfrak{g}$ generated by the simple reflections $\{r_i | i \in I\}$. For $\alpha, \beta \in Q$, we define $\alpha > \beta$ if $\alpha - \beta \in Q_+ = \sum Z_{\geq 0}\alpha_i$. For $\alpha = \sum_{i=0}^{l} k_i\alpha_i \in Q$, we define $\mathrm{ht}(\alpha) = \sum_{i=0}^{l} k_i$ to be the height of α.

It is known that the generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ is either (i) finite (ii) affine or (iii) indefinite type Kac [3, Theorem 4.3].

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In Billig & Pianzola [5], Billig and Pianzola conjectured that the nilpotency index of the subalgebra S_w is bounded by a constant $k = k(A)$ which depends only on the Cartan matrix A not on w. We (Kim, Misra, Stizinger) settled this conjecture in most case Kim, Misra & Stitzinger [4]. This information helps us in constructing a sequence of root spaces with certain conditions.

In this paper, we study the root system of affine Lie algebra of type $D_l^{(1)}$. We construct a sequence of root spaces with some conditions. And then we prove that the number of this sequence is finite (less than or equal to the height of θ, the highest root.)

1. Root system of Affine Lie algebra of type $D_l^{(1)}$

Let $A = D_l^{(1)}$ and $\mathfrak{g} = \mathfrak{g}(A)$ be the associated affine Lie algebra. Let $\mathring{\mathfrak{g}} = \mathfrak{g}(\mathring{A})$ be the corresponding simple Lie algebra with Carten matrix $\mathring{A} = (a_{ij})_{i,j=1}^{l}$. Let Δ and $\mathring{\Delta}$ denote the set of roots for $\mathfrak{g}$ and $\mathring{\mathfrak{g}}$ respectively. Then $\Delta = \Delta^{re} \cup \Delta^{im}$, where Δ^{re}, Δ^{im} denote the real, imaginary roots respectively. It is known that the set of positive imaginary roots $\Delta_+^{im} = \{n\delta \mid n \in \mathbb{Z}_{>0}\}$, where $\delta = \sum_{i=0}^{l} a_i\alpha_i$, $a_i \in \mathbb{Z}_{>0}$, $gcd(a_0, \ldots, a_n) = 1$ and $A(a_0, \ldots, a_n)^T = 0$. Note that $\Delta = \Delta_+ \cup \Delta_-$, $\mathring{\Delta} = \mathring{\Delta}_+ \cup \mathring{\Delta}_-$, $\Delta^{re} = \Delta_+^{re} \cup \Delta_-^{re}$, $\Delta^{im} = \Delta_+^{im} \cup \Delta_-^{im}$, where the subscript plus/minus denote the positive/negative roots. Let $\mathring{\Pi} = \{\alpha_1, \ldots, \alpha_l\}$ and $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ denote the simple roots for $\mathring{\mathfrak{g}}$ and g respectively.

Proposition 1.1 (Humphreys [1]). *Each $\alpha \in \Delta_+$ can be written in the form $\alpha_{m_1} + \alpha_{m_2} + \cdots + \alpha_{m_k}$ ($\alpha_{m_i} \in \Pi$, not necessarily distinct) in such a way that each partial sum $\alpha_{m_1} + \alpha_{m_2} + \cdots + \alpha_{m_i}$ is a root.*

Corollary 1.2. *Let $\mathfrak{g} = \mathfrak{g}(A)$ be a Lie algebra of type $D_l^{(1)}$. Let $\theta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l$, the highest root in $\mathring{\Delta}_+$. Then there is a sequence $\{\theta_n\}$ in $\mathring{\Delta}_+$ such that $ht(\theta_n) = n$ for $n = 1, 2, \ldots, 2l-3$ and $\theta_{2l-3} = \theta$.*

Note that $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l$ and $\delta - \alpha_0 = \theta$ Kac [3]. The following proposition describes the set of real roots Δ^{re} in terms of $\mathring{\Delta}$ and δ.

Proposition 1.3 (Kac [3]). *Let $\mathfrak{g} = \mathfrak{g}(A)$ be a Lie algebra of type $D_l^{(1)}$. Then $\Delta^{re} = \{\alpha + n\delta \mid \alpha \in \mathring{\Delta},\ n \in \mathbb{Z}\}$.*

Let $\mathring{W}$and W denote the Weyl groups of $\mathring{\mathfrak{g}}$ and g, respectively. Note that the set of imaginary roots $\Delta_{\pm}^{im}$ are invariant under the action of Weyl group W Kac [3].

Furthermore, for $w \in W$, we have

$$\Delta^+(w) = \{\alpha \in \Delta_+ \mid w^{-1}(\alpha) < 0\} \subset \Delta_+^{re}.$$

Let $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_t} \in \Pi$ (not necessarily distinct) and denote r_i the simple reflection of W. When $w \in W$ is written as $w = r_{i_1} \cdots r_{i_t}$ ($\alpha_{i_j} \in \Pi$, t minimal), we call the expression reduced. We call t the length of w and is denoted by $l(w)$.

Proposition 1.4 (Wan [6]). *Let $w = r_{i_1} \cdots r_{i_t} \in W$ be a reduced expression of w. Then*

$$\Delta^+(w) = \{\beta_1, \ldots, \beta_t\},$$

where $\beta_p = r_{i_1} \cdots r_{i_{p-1}}(\alpha_{i_p})\,(1 \le p \le t)$ and the β_p are all distinct. In particular, $l(w) = |\Delta^+(w)|$.

Note that $\mathring{\Pi}$ and $-\mathring{\Pi}$ are root bases of Δ. Since $\mathring{W}$ acts transitively on the bases Humphreys [1], there exists $w_0 \in \mathring{W}$ such that $w_0^{-1}(\mathring{\Pi}) = -\mathring{\Pi}$

Proposition 1.5. *Let $\mathfrak{g} = \mathfrak{g}(A)$ be an affine Lie algebra of type $D_l^{(1)}$. Let*

$$\begin{aligned} w_0 = {} & (r_1 r_2 r_3 \cdots r_{l-3} r_{l-2} r_l r_{l-1} r_{l-2} r_{l-3} \cdots r_3 r_2 r_1) \\ & \times (r_2 r_3 \cdots r_{l-3} r_{l-2} r_l r_{l-1} r_{l-2} r_{l-3} \cdots r_3 r_2) \\ & \times (r_3 \cdots r_{l-3} r_{l-2} r_l r_{l-1} r_{l-2} r_{l-3} \cdots r_3) \cdots (r_{l-3} r_{l-2} r_l r_{l-1} r_{l-2} r_{l-3}) \\ & \times (r_{l-2} r_l r_{l-1} r_{l-2})(r_l r_{l-1}). \end{aligned}$$

Then $w_0^{-1}(\alpha_i) = -\alpha_i$ for $i = 1, 2, \ldots, n$. Furthermore, $\Delta^+(w_0) = \mathring{\Delta}_+$.

Proof. By simple calculation, $w_0^{-1}(\alpha_i) = -\alpha_i$ for $i = 1, 2, \ldots, n$ and hence

$$\mathring{\Delta}_+ \subseteq \Delta^+(w_0).$$

Since $|\Delta^+(w_0)| = l(w_0) = l(l-1) = |\mathring{\Delta}_+|$, we get a desired result. □

2. Main results

Let $\mathfrak{g} = \mathfrak{g}(A)$ be a Lie algebra of type $D_l^{(1)}$ and θ be the highest long root in $\mathring{\Delta}_+$. Construct a sequence $\{\theta_n\}$ as follows:

$$\theta_n = \begin{cases} \sum_{i=1}^{n} \alpha_i & \text{for } 1 \le n \le l-2, \\ \sum_{i=1}^{l-2} \alpha_i + \alpha_{l-1} \text{ or } \sum_{i=1}^{l-2} \alpha_i + \alpha_l & \text{for } n = l-1, \\ \sum_{i=1}^{l} \alpha_i & \text{for } n = l, \\ \sum_{i=1}^{l} \alpha_i + \sum_{i=2l-n-1}^{l-2} \alpha_i & \text{for } l < n \le 2l-3, \end{cases} \tag{2-1}$$

Then sequence $\{\theta_n\}$ has the following properties:

(a) $\{\theta_n\}$ is a sequence in $\mathring{\Delta}_+(D_l^{(1)})$ with $\mathrm{ht}(\theta_n) = n$ for $n = 1, 2, \ldots, 2l-3$.

(b) $\beta \in \mathring{\Delta}$ implies $\beta = \theta_j - \theta_i$ for some $1 \leq i, j \leq 2l-3$.

In the above sequence $\{\theta_n\}$, let

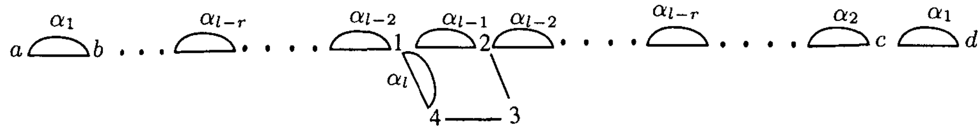
$$\sum_{i=1}^{l-2} \alpha_i + \alpha_{l-1} = \theta_{l-1}, \sum_{i=1}^{l-2} \alpha_i + \alpha_l = \theta_{\overline{l-1}}.$$

Then $\alpha_l = \theta_{\overline{l-1}} - \theta_{l-2}$, $\alpha_{l-1} = \theta_{l-1} - \theta_{l-2}$.

Let $\theta_0 = 0$, $\theta_{2l-2} = \theta + \alpha_1$. For $i \geq 2$, α_{l-i} can be represented by two ways, $\alpha_{l-i} = \theta_{l-i} - \theta_{l-i-1} = \theta_{l+i-1} - \theta_{l+i-2}$.

Draw a line and place vertices corresponding to θ_n for $n = 0, 1, \ldots 2l-2$ as follows. Let a, b, c and d stand for $\theta_0, \theta_1, \theta_{2l-3}$ and θ_{2l-2} respectively. Let $1, 2, 3$ and 4 stand for $\theta_{l-2}, \theta_{l-1}, \theta_l$ and $\theta_{\overline{l-1}}$ respectively.

(2-2)

$D_l^{(1)}$ 

As was shown in the diagram, we identify α_i as the segment connecting two adjacent vertices.

Now for $w \in W$, we define

$$X = \{\alpha \in \mathring{\Delta}_+ \mid w^{-1}(\alpha + n\delta) < 0 \ \text{ for some } \ n \in \mathbb{Z}_{\geq 0}\},$$

and

$$Y = \{\alpha \in \mathring{\Delta}_- \mid w^{-1}(\alpha + n\delta) < 0 \ \text{ for some } \ n \in \mathbb{Z}_{>0}\}.$$

We denote $-X = \{ -\alpha \mid \alpha \in X \}$ and $-Y = \{ -\alpha \mid \alpha \in Y \}$.

Construct a sequence $\{\beta_n\}$ in $X \cup Y$ such that for each n,

(2-3) $$s_n = \sum_{i=1}^{n} \beta_i \in X \cup Y \text{ where } \beta_i \in X \cup Y$$

are not necessarily distinct. We have the following Proposition.

Proposition 2.1 (Kim, Misra & Stitzinger [4]). *Let $\mathfrak{g} = \mathfrak{g}(A)$ be an affine Lie algebra with W as its Weyl group and let w be any element of W, and $\{s_n\}$ be the sequence constructed as in* (2-3). *Then no partial sum of the subsequence $\{\beta_{i_n}\}$ of $\{\beta_n\}$ is equal to* 0.

On the diagram of $D_l^{(1)}$, let us identify $\beta = \theta_j - \theta_i$ with bending line connecting two vertices θ_i and θ_j. Let bending line above and below stand for positive and negative root, respectively. Then $s_n = \sum_{i=1}^{n} \beta_i$ can be represented as the connected n numbers of bending lines. In other words, we can see the process of addition of $\beta_i's$ on the diagram. We will say that the diagram of $\{s_n\}$ has no cycle if each vertex θ_i meets at most once with bending lines which represent $\{s_n\}$.

Corollary 2.2. *The diagram of the sequence $\{s_n\}$ has no cycle.*

Draw the diagram of the sequence $\{s_n\}$ in (2-2). Whenever we add

(2-4) $\beta_i = \theta_k - \theta_j$, place the asterix $(*)$ sign on $\begin{cases} \text{(a) the vertex } \theta_k \text{ for } j < k, \\ \text{(b) the vertex } \theta_j \text{ for } j > k. \end{cases}$

As was shown in above, $s_n = \sum_{i=1}^{n} \beta_i$ can be represented as the connected n numbers of bending lines which has two end vertices, one is marked by $*$ and the other is not marked. Call the vertex which is not marked the initial vertex and the other the terminal vertex. Reorder the vertex which are marked by $*$ from the initial vertex to terminal vertex following the connected bending lines. If n_1, n_2, n_3, n_4 were marked with $*$ in this order, then we write $n_1n_2n_3n_4$. Note that xy means y follows after x but not necessarily immediately.

We identify $xy = y - x$. In other words, we can identify xy with some element $\alpha = \sum_{i=p}^{q} k_i\alpha_i \in Q$. By this identification, $xy = -yx$. Let x, y, s, t are vertices on the diagram. If $xy + st = 0$, then xy and st makes a cycle.

Lemma 2.3. *Let A is of type $D_l^{(1)}$ and $\mathfrak{g} = \mathfrak{g}(A)$ is the associated Lie algebra. Let a,b,c,d are vertices in* (2-2). *Then there are 8 possible enumerations of a,b,c,d.*

Proof. Since $ab + dc = ac + db = 0$, (ab and dc) and (ac and db) make cycles, a contradiction. Similarly, (ba and cd) and (ca and bd) make cycles, a contradiction.

Therefore, the remaining possibilities are;

abcd, acbd, badc, bdac, cadb, cdab, dcba, dbca, we are done. □

Lemma 2.4. *Let A is of type $D_l^{(1)}$ and $\mathfrak{g} = \mathfrak{g}(A)$ is the associated Lie algebra. Then there are* 8 *possible enumerations of* 1, 2, 3, 4 *in* (2-2).

Proof. Since $12 + 34 = 14 + 32 = 0$, (12 and 34) and (14 and 32) make cycle, a contraction. Similarly, (21 and 43) and (32 and 14) make cycle, a contraction.

Therefore, the remaining possibilities are; 1243, 1423, 2134, 2314, 3241, 3421, 4132, 4312, we are done. □

In the case when a is the initial vertex and b, c, d are dotted vertex on the diagram marked in this order, we also write $abcd$.

Lemma 2.5. *Let A is of type $D_l^{(1)}$ and $\mathfrak{g} = \mathfrak{g}(A)$ is the associated Lie algebra and let $n_1n_2n_3n_4$ be one of the 8 possibilities in the Lemma 2.4. Then n_3 cannot follow immediately after n_2.*

Proof. Note that the possibilities for n_2n_3 are 24, 42, 13, or 31. Suppose that there is no vertex marked between n_2 and n_3. Then $\alpha_{l-1} - \alpha_l$, $-\alpha_{l-1} + \alpha_l$, $\alpha_{l-1} + \alpha_l$, or $-\alpha_{l-1} - \alpha_l$ is a root, a contradiction. □

Let x, y be elements on the diagram. We will say that two different vertices x and y are sterile if $x - y$ is not a root. We know θ_i and θ_j are sterile if $i + j = 2l - 2$ or $(i, j) = (l - 1, \overline{l - 1})$.

We have the following Lemma.

Lemma 2.6. *If two vertices x and y are sterile, then x and y are not next immediately to each other.*

Proof. Suppose that x and y are adjacent vertices. Then $x - y$ is a root, a contradiction. □

Lemma 2.7. *Let $s_1s_2s_3s_4$ be one of the 8 possible enumerations of a,b,c,d in the Lemma 2.3 and let $n_1n_2n_3n_4$ be one of the 8 possible enumerations of $1, 2, 3, 4$ in the Lemma 2.4. Then there are 6 possible enumerations of $1, 2, 3, 4$ and a, b, c, d. Namely,*

$$s_1s_2n_1n_2n_3n_4s_3s_4, \quad s_1n_1s_2n_2n_3s_3n_4s_4, \quad s_1n_1n_2s_2s_3n_3n_4s_4,$$

$$n_1n_2s_1s_2s_3s_4n_3n_4, \quad n_1s_1n_2s_2s_3n_3s_4n_4, \quad n_1s_1s_2n_2n_3s_3s_4n_4.$$

Proof. Since $1a + 3d = 1b + 3c = 4a + 2d = 2a + 4d = 4b + 2c = 2b + 4c = 3b + 1c = 3a + 1d = 0$ and $s_1n_1 + s_4n_4 = s_1n_4 + s_4n_1 = s_1n_2 + s_4n_3 = s_1n_3 + s_4n_2 = s_2n_1 + s_3n_4 = s_2n_4 + s_3n_1 = s_2n_2 + s_3n_3 = s_2n_3 + s_3n_2 = 0$. By Corollary 2.2, only 6 enumerations are permitted, we are done. □

Proposition 2.8. *There exist at least two vertices on the diagram which are not marked.*

Proof. Suppose that n_1 is the initial vertex and n_4 is the terminal vertex, then one of $\pm\alpha_{l-1} \pm \alpha_l$, is a root, a contradiction. If s_1 is the initial vertex and s_4 is the terminal vertex, then one of $2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_l$ or $-(2\alpha_2 + \cdots + 2\alpha_l)$ is a root, a contradiction.

Consider the 6 possible enumerations in Lemma 2.7. In case of $l = 4$, there is no vertex on the diagram different from $1, 2, 3, 4$ and a, b, c, d. If s_1 is the initial vertex, then (n_3 and s_4) or (s_3 and s_4) cannot be marked. In the case when n_1 is the initial vertex, (s_3 and n_4) or (n_3 and s_4) cannot be marked, we are done.

Assume that $l > 4$. Among the 6 enumerations, consider the case

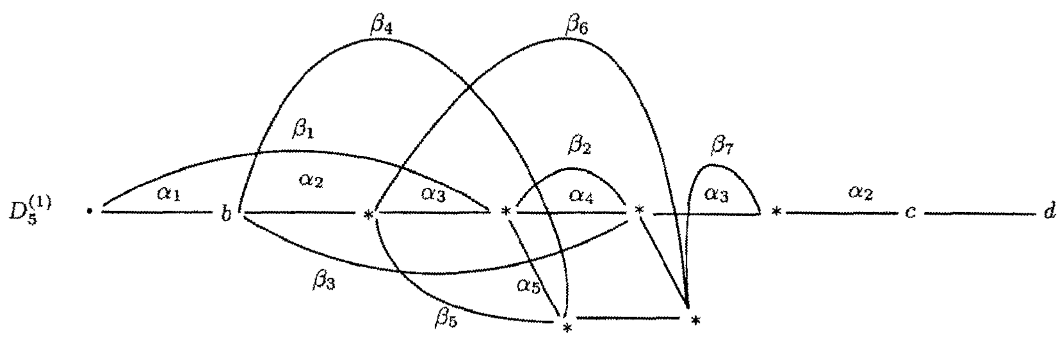
$$s_1 s_2 n_1 n_2 n_3 n_4 s_3 s_4.$$

Since s_1 and s_4 are sterile, there exists a vertex x_1 with $x_1 s_1$ on the diagram different from a, b, c, d, 1, 2, 3 and 4. Let y_1 be the sterile vertex with x_1. Then we have 10 enumerations as follows:

$$y_1 x_1 s_1 s_2 n_1 n_2 n_3 n_4 s_3 s_4,\ x_1 y_1 s_1 s_2 n_1 n_2 n_3 n_4 s_3 s_4,\ x_1 s_1 y_1 s_2 n_1 n_2 n_3 n_4 s_3 s_4,$$
$$x_1 s_1 s_2 y_1 n_1 n_2 n_3 n_4 s_3 s_4,\ x_1 s_1 s_2 n_1 y_1 n_2 n_3 n_4 s_3 s_4,\ x_1 s_1 s_2 n_1 n_2 y_1 n_3 n_4 s_3 s_4,$$
$$x_1 s_1 s_2 n_1 n_2 n_3 y_1 n_4 s_3 s_4,\ x_1 s_1 s_2 n_1 n_2 n_3 n_4 y_1 s_3 s_4,\ x_1 s_1 s_2 n_1 n_2 n_3 n_4 s_3 y_1 s_4,$$
$$x_1 s_1 s_2 n_1 n_2 n_3 n_4 s_3 s_4 y_1.$$

Since $x_1 s_1 + y_1 s_4 = 0$, y_1 cannot be marked or there exists a vertex x_2 with $x_2 x_1$. We can repeat this process only finite times. Thus we can find x_n with $x_n s_1$ such that its sterile vertex y_n cannot be marked. On the other hand, there exists u_1 with $n_2 u_1 n_3$. Then its sterile vertex v_1 cannot be marked or there exists u_2 with $n_2 u_2 u_1 n_3$ or $n_2 u_1 u_2 n_3$. After finitely many steps, we can find u_t such that its sterile vertex cannot be marked, we are done.

Remaining cases are treated in the same manner. □



Example. In $D_5^{(1)}$, $\theta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5$, and hence $ht(\theta) = 7$. $\beta_1 = \alpha_1 + \alpha_2 + \alpha_3$, $\beta_2 = \alpha_4$, $\beta_3 = -\alpha_2 - \alpha_3 - \alpha_4$, $\beta_4 = \alpha_2 + \alpha_3 + \alpha_5$, $\beta_5 = -\alpha_3 - \alpha_5$, $\beta_6 = \alpha_3 + \alpha_4 + \alpha_5, \beta_7 = \alpha_3$. The process can go no further. If $\beta_8 = \alpha_2$, then $\beta_2 + \beta_3 + \beta_7 + \beta_8 = 0$, a contradiction. If $\beta_8 = -\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5$, then $\beta_1 + \beta_2 + \beta_4 + \beta_8 = 0$, a contraction. Thus c and d cannot be marked.

Proposition 2.9. *Let A be of type $D_l^{(1)}$ and $\mathfrak{g} = \mathfrak{g}(A)$ be the associated Lie algebra and let $\{s_n\}$ the sequence constructed in* (2-3). *Then the number of terms of the sequence $\{s_n\}$ cannot exceed $2l - 3$, the height of θ.*

Proof. In the diagram in (2-3), the bending line of $s_1 = \beta_1$ uses two vertices, $s_2 = \beta_1 + \beta_2$ uses 2+1 vertices. Thus s_t uses $t + 1$ vertices on the diagram. There are $2l$ vertices on the diagram in (2-3) and one is the initial vertex. Since no vertex can be used twice by Proposition 2.3, Proposition 2.8 shows that the maximum number to which the addition is allowed is also $2l - 1 - 2$, we are done. □

Let w be an element of W.

(2-5) Construct a sequence

$$\{\mathfrak{g}_{\gamma_k}\} \text{ in } \Delta^+(w) \text{ and } \gamma_{k+1} = \gamma_k + \gamma \text{ for some } \gamma, \gamma_k \in \Delta^+(w).$$

We have the following Theorem.

Theorem 2.10. *The number of terms of the sequence $\{\mathfrak{g}_{\gamma_k}\}$ in* (2-3) *cannot exceed $2l - 3$, the height of θ.*

Proof. Let $\gamma_k = \beta_k + n_k\delta$ for $\beta_k \in X \cup Y$, $k - 1, 2, \ldots$. Then we have the sequence $\{\beta_k\}$ in $X \cup Y$ such that $\beta_{k+1} = \beta_k + \beta$ for some $\beta \in X \cup Y$. But the number of sequence $\{\beta_k\}$ cannot exceed $2l - 3$ by Proposition 2.9 , we are done. □

Corollary 2.11. *There exists a sequence $\{\mathfrak{g}_{\gamma_k}\}$ satisfying the conditions in* (2-3) *which has exactly $2l - 3$ terms.*

Proof. Let $w = w_0$ as in Proposition 1.5 and let $\gamma_k = \theta_k$. Since the sequence θ_k has $2l - 3$ terms and $\Delta^+(w_0) = \mathring{\Delta}_+$, the sequence $\{\mathfrak{g}_{\theta_k}\}$ is a desired sequence. □

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