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A SURVEY OF BLOCH CONSTANTS

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1. Introduction

We begin with a brief survey of some of the known results dealing with Bloch constants. Bloch's theorem asserts that there is a constant $B_{1.C}(1,0)$ such that if f is holomorphic in the open unit disk D and normalized by $|f'(0)| \geq 1$, then the Riemann surface of f contains an unramified disk of radius at least $B_{1.C}(1,0)$ (see[7,p.14]). Pommerenke[6] introduced the locally schlicht Bloch constant

$$B_{\infty.C}(1,0) > B_{1.C}(1,0).$$

The classical notion of Landau constant applies to holomorphic mappings of D into the complex plane C. Then we relate these Landau constant to the Bloch constants. We know that $B_{\infty.C}(1,0) \leq L_C(1,0)$, where $L_C(1,0)$ denotes the Landau constant. The exact value of these constants is still unknown, but the following bounds have been established:

$$0.433 < \frac{\sqrt{3}}{4} < \frac{\sqrt{3}}{4} + 10^{-4} < B_{1.C}(1,0) \leq \frac{1}{\sqrt{1+\sqrt{3}}} \frac{\Gamma(1/3)\Gamma(11/12)}{\Gamma(1/4)} < 0.4719$$

$$0.5 < B_{\infty.C}(1,0) \leq L_C(1,0) \leq \frac{\Gamma(1/3)\Gamma(5/6)}{\Gamma(1/6)} < 0.5433,$$

where Γ denotes the Gamma function. Basic material and associated distance function that is required for the definition of Bloch constants and Landau constant is presented in Section 2. In Section 3 we exhibit a unified approach to obtaining upper and lower bounds for other Bloch constants. Section 4 is devoted to the Landau constant and analogs. In Section 5 we investigate that the Bloch constant

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX for the family of all non-constant meromorpic functions on C lies between $\frac{\pi}{3}$ and $2\arctan(\frac{1}{\sqrt{2}})$. In addition,we give that the Bloch constant for the family of locally schlicht meromorphic functions on C is $\frac{\pi}{2}$.

2. Notations and terminology

Let Ω be Riemann surface. For $m \in Z^+$, let $F_m(D,\Omega)$ denote the family of all analytic function $f : D \to \Omega$ such that any $q \in f(D)$ each root of $f(p) = q$ is either simple or has multiplicity at least $m+1$. The family of all analytic functions $f : D \to \Omega$ is simple is denoted by $F_\infty(D,\Omega)$. We view R_f, the Riemann surface of f, as being spread over Ω. In order to measure the size of unramified disks on R_f,we need to fix a distance function on Ω, where Ω is one of D, C and the Riemann sphere P. Let

$$\delta_D(z,w) = |\frac{z-w}{1-\overline{w}z}| \qquad \text{if } z,w \in D,$$
$$\delta_C(z,w) = |z-w| \qquad \text{if } z,w \in C,$$
$$\delta_P(z,w) = |\frac{z-w}{1+\overline{w}z}| \qquad \text{if } z,w \in P.$$

Let $f \in F_m(D,\Omega), D_\Omega(a,r) = \{z \in \Omega : \delta_\Omega(a,z) < r\}$ is called the open disk in Ω with center a and radius r. For $p \in D$ set

$$r_\Omega(p,f) = \sup\ \{r : D_\Omega(f(p),r) \text{ is an unramified disk contained in } R_f\}.$$

Also, define

$$r_\Omega(f) = \sup\{r_\Omega(p,f) : p \in D\}.$$

Set

$$f^D(z) = \frac{(1-|z|^2)|f'(z)|}{1-|f'(z)|^2},$$
$$f^C(z) = (1-|z|^2)|f'(z)|,$$
$$f^P(z) = \frac{(1-|z|^2)|f'(z)|}{1+|f'(z)|^2}.$$

Now we can introduce the various Bloch constants. For $p \in D$ and $\alpha > 0$, let

$$B_{m.\Omega}(\alpha, p) = \inf \{r_\Omega(f) : f \in F_m(D, \Omega) \text{ and } f^\Omega(p) \geq \alpha\}.$$

Observe that $B_{1.C}(1,0)$ is the classical Bloch constant and $B_{\infty.C}(1,0)$ is the locally schlicht Bloch constant.

3.A bound for Bloch costants

Before establishing a lower bound for the Bloch constants, we introduce another function. Let

$$\phi_m(t) = \begin{cases} \frac{R^{1/m+1}}{(m+1)t^{m/m+1}(R^{2/m+1} - t^{2/m+1})}, & m \in Z^+, \\ \frac{1}{2t\log(R/t)}, & m = \infty. \end{cases}$$

Given $s > 0$,

$$R_{m.C}(s) = \begin{cases} s(\frac{m+2}{m})^{(m+1)/2}, & m \in Z^+, \\ se, & m = \infty. \end{cases}$$

Let $\phi_{m.s}$ denote the function ϕ_m with R replaced by $R_{m.C}(s)$. The minimum value of $\phi_{m.s}$ on the interval $(0, R_{m.C}(s))$ is

$$\phi_{m.s}(s) = \begin{cases} \frac{[m(m+2)]^{1/2}}{2(m+1)s}, & m \in Z^+, \\ \frac{1}{2s}, & m = \infty. \end{cases}$$

Let $h_{m.C}(s) = \frac{1}{\phi_{m.s}(s)}$ $m \in Z^+ \cup \{\infty\}$. Then $h_{m.C}$ is strictly increasing on $(0, \infty)$. The inverse function is given by

$$h^{-1}_{m.C}(\alpha) = \begin{cases} \alpha[m(m+2)]^{1/2}, & m \in Z^+, \\ \frac{\alpha}{2}, & m = \infty. \end{cases}$$

The following distortion theorem is a result of D.Minda [9].

Theorem 3.1. Let $f \in F_m(D, C)$. If $r_C(f) \leq s$, then for $p \in D$

$$f^C(p) < \frac{1}{\phi_{m.s}(r_C(p, f))}.$$

Theorem 3.2. Let $p \in D$. Then $h_{m.C}^{-1}(\alpha) < B_{m.C}(\alpha, p)$.

Proof. If $B_{m.C}(\alpha, p) = \infty$, then there is nothing to prove. Consequently, we may assume that $B_{m.C}(\alpha, p) < \infty$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $F_m(D, C)$ such that $f_n^C(p) \geq \alpha$ and $r_C(f_n) < B_{m.C}(\alpha, p) + \frac{1}{n} = s_n$. Without loss of generality we may assume that $f_n(p) = 0$ for all n. Since $r_C(f_n) < s_1$ for all n, Theorem 3.1 implies that for any $q \in D$

$$f_n^C(q) < \frac{1}{\phi_{m.s_1}(r_C(q, f_n))} < \frac{1}{\phi_{m.s_1}(s_1)} \qquad \text{for all } n.$$

Then $\{f_n\}_{n=1}^{\infty}$ is a normal family, so we may assume that $f_n \to f$ locally uniformly on D. Hurwitz's theorem implies that $f \in F_m(D, C)$. Clearly, $f(p) = 0$ and $f^C(p) \geq \alpha$ and $r_C(f) = B_{m.C}(\alpha, p) = s$. Now Theorem 3.1 applied to the function f at the point p gives

$$\alpha \leq f^C(p) < h_{m.C}(s).$$

Since $h_{m.C}$ is incerasing, we obtain a contradiction if $s \leq h_{m.C}^{-1}(\alpha)$; hence $h_{m.C}^{-1}(\alpha) < B_{m.C}(\alpha, p)$. □

For $\alpha = 1$ and $m = 1, \infty$ we obtain $\frac{\sqrt{3}}{4} < B_{1.C}(1, p)$, $\frac{1}{2} < B_{\infty.C}(1, p)$. Ahlfors[2] and Heins[3] established the lower bound $\frac{\sqrt{3}}{4} < B_{1.C}(1, p)$. In 1988 Bonk[11] improved the lower bound on the Bloch constant to $B_{1.C}(1, p) > \frac{\sqrt{3}}{4} + 10^{-14}$. The inequality $\frac{1}{2} < B_{\infty.C}(1, p)$ was established by Ahlfors[2] and Pommerenke[6].

We know a upper bound for the Bloch constants by modifying the example of Ahlfors and Grunsky[1]. We determined of $R_q, q \in [0, \frac{5}{3})$ as follow. For $q \in [0, \frac{1}{3})$ we require that

$$R_q = [\frac{\sin\pi(5/6 + q/2)}{\sin\pi(1/6 + q/2)}]^{1/2}, \qquad q \in [0, 1/3).$$

And we demands that

$$R_q = [\frac{-\sin\pi(5/6 + q/2)}{\sin\pi(1/6 + q/2)}]^{1/2}, \qquad q \in [1/3, 5/3).$$

Then [4,p.367]

$$f'_q(0) = \frac{\Gamma(5/6 + q/2)\Gamma(2/3)}{\Gamma(1/6 + q/2)\Gamma(4/3)}.$$

Define $g_{m.q}(z) = R_q f_q \circ f^{-1}_{q/m+1}(\frac{z}{R_{q/m+1}})$ for $m \in Z^+ \cup \{\infty\}$ and $q \in [0, 5/3)$. We obtain

$$\alpha_{m.q} = g'_{m.q}(0) = \frac{R_q \Gamma(5/6 + q/2)\Gamma(1/6 + q/2(m+1))}{R_{q/(m+1)}\Gamma(1/6 + q/2)\Gamma(6/5 + q/2(m+1))}.$$

We consider the functions $g_{m.1/3}$ where $m \in Z^+ \cup \{\infty\}$. Then

$$g_{m.1/3} \in F_m(D.C) \text{ and } r_C(g_{m.1/3}) = 1 = R_{1/3}.$$

This gives

$$\alpha_{m.1/3} B_{m.C}(1,0) = B_{m.C}(\alpha_{m.1/3}, 0) \leq 1$$

or

$$B_{m.C}(1,0) \leq \frac{1}{\alpha_{m.1/3}}.$$

Thus we know

$$B_{1.C}(1,0) \leq \frac{1}{\sqrt{1+\sqrt{3}}} \frac{\Gamma(1/3)\Gamma(11/12)}{\Gamma(1/4)} < 0.4719,$$

$$B_{\infty.C}(1,0) \leq \frac{\Gamma(1/3)\Gamma(5/6)}{\Gamma(1/6)} < 0.5433.$$

In [4,p.368] the same method is used to obtain an upper bound for $B_{\infty.C}(1,0)$. Pommerenke [6] cites 0.555 at the best known value of the locally schlicht Bloch constant. Consequently, this value of 0.5433 is the best known value of the locally schlicht Bloch constant.

4. Landau costant

The classical notion of Landau constant applies to holomorphic mapping of D into C. Let $R_C(p,f)$ is the radius of the largest disk centered at $f(p)$ which is contained in $f(D)$,

$$R_C(f) = \sup\{R_C(p,f) : p \in D\}.$$

For $p \in X$ and $\alpha > 0$, let

$$L_C(\alpha,p) = \inf\{R_C(f) : f \in F_1(D,C) \text{and} f^C(p) \geq \alpha\}.$$

Then $L_C(1,0)$ is the classical Landau constant. Clearly, $r_C(p,f) \leq R_C(p,f)$. Note that $B_{1.C}(\alpha,p) \leq L_C(\alpha,p)$.

Theorem 4.1. For $p \in D$. Then $h_{\infty.C}^{-1} < L_C(\alpha,p)$.

Proof. The proof of Theorem 4.1 follows just like the proof of Theorem 3.1.

By([9.Theorem 12]), we can show that the following Theorem 4.2

Theorem 4.2. $B_{\infty.C}(1,0) \leq L_C(1,0)$.

We consider the function $g_{\infty.\frac{1}{n}}, n \in Z^+$, that were constructed in Section 3. In the notation of Section 3 we obtain $L_C(\alpha_{\infty.\frac{1}{n}},0) \leq R_{1/3}$. In particular, for $n = 3$ we get

$$L_C(1,0) \leq \frac{\Gamma(1/3)\Gamma(5/6)}{\Gamma(1/6)} < 0.5433.$$

So, we have $B_{\infty.C}(1,0) \leq L_C(1,0) < 0.5433$. This unpublished example and bound was cited in a footnote to [2].

5. A bound for the Bloch constants for the meromorphic functions

Let P denote the Riemann sphere. These results have applications to meromorphic functions defined on C. Let $G_m(C,P)$ denote the family of all nonconstant meromorphic functions $f : C \to P$ such that for each $q \in f(C)$ each root of $f = q$ is either simple or else has multiplicity at lest $m+1$. For $m = \infty$ this gives locally schlicht meromorphic functions. For $p \in C$ let $r_P(p,f)$ denote the radius of the largest unramified disk in R_f with center$f(p)$. Also, let $r_P(f) = \sup\{r_P(p,f) : p \in C\}$. Let

$$E_{m.P} = \inf\{r_P(f) : f \in G_m(C,P)\}.$$

Now, we have a lower bound for the various Bloch constants $E_{m.P}$.

Theorem 5.1. $E_{m.P} \geq 2\tan^{-1}(\sqrt{\frac{m}{m+2}})$ for $m \in Z^+ \cup \{\infty\}$.

Proof. We shall assumption that $f \in G_m(C, P)$ and $r_P(f) < 2\tan^{-1}(\sqrt{\frac{m}{m+2}})$. Suppose X is a hyperbolic region in C and $\lambda_X(z)|dz|$ is the hyperbolic metric on X which has constant curvature -1. If $f \in G_m(C, P)$ and $r_P(f) \leq 2\tan^{-1}(s)$ where $s < \sqrt{\frac{m}{m+2}}$, then for $p \in X$,

$$\frac{2|f'(p)|}{1+|f(p)|^2}\frac{1}{\lambda_X(p)} < \frac{1}{\psi_m(s)},$$

Where

$$\psi_m(s) = \begin{cases} (m+2-ms^2)^{1/2}(m-(m+2)s^2)^{1/2}, & m \in Z^+ \\ \frac{1-s^2}{2s}, & m = \infty. \end{cases}$$

This result is due to Minda[9] for X an arbitrary hyperbolic Riemann surface. Then $f \in G_m(B(a, R), P)$ for any $R > 0$, where $B(a, R) = \{z : |z-a| < R\}$. The hyperbolic metric for $B(a, R)$ is $\frac{2R|dz|}{R^2-|z-a|^2}$. Then we obtain

$$\frac{|f'(a)|}{1+|f(a)|^2} < \frac{1}{R\psi_m(s)}.$$

If we let $R \to \infty$, then we get $f'(a) = 0$. Thus f is constant, which is a contradiction. □

For $m = 1$ we obtain

$$E_{1.P} \geq \tfrac{\pi}{3} \text{ and } E_{\infty.P} \geq \tfrac{\pi}{2}.$$

Now we give a upper bound for $E_{m.P}$ in case $m = 1, \infty$. The exponential function is meromorphic on C. Since the image can never contain a spherical disk which is a larger than a hemisphere. Thus $r_P(\exp) = \frac{\pi}{2}$. This implies that $E_{\infty.P} \leq \frac{\pi}{2}$ since exp is locally schlicht. Next we know that

$$E_{1.P} \leq 2\arctan(\frac{1}{\sqrt{2}}).$$

Ahlfors and Grunsky[1] gave an upper bound for the classical Bloch constant. By a natural extension of the idea involved in the construction of the Ahlfors-Grunsky example. Let $\triangle_{1/3}$ be the euclidean triangle with $1, \omega$ and ω^2 where $\omega = \exp(\frac{2\pi i}{3})$. Let $\triangle_{2/3}$ be the regular circular triangle which has all interior angles of size $\frac{2\pi}{3}$ and vertices at the point $\frac{1}{\sqrt{2}}, \frac{\omega}{\sqrt{2}}$ and $\frac{\omega^2}{\sqrt{2}}$. Let g be the unique conformal mapping $\triangle_{1/3}$on $\triangle_{2/3}$ which satisfies the condition $g(\omega^j) = \frac{\omega^j}{\sqrt{2}}$. Then $g \in G_1(C, P)$ and $r_P(g) = 2\arctan(\frac{1}{\sqrt{2}})$. Consequently

$$E_{1.P} \leq 2\arctan(\frac{1}{\sqrt{2}}).$$

So we have

$$E_{\infty.P} = \frac{\pi}{2}$$

$$\frac{\pi}{3} \leq E_{1.P} \leq 2\arctan(\frac{1}{\sqrt{2}}).$$

We also obtain results for Bloch constants for meromorphic functions on compact Riemann surfaces in [8].

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