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BASIS FOR ALMOST LINEAR SPACES

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ABSTRACT. In this paper, we introduce the almost linear spaces, a generalization of linear spaces. We prove that if the almost linear space X has a finite basis then, as in the case of a linear space, the cardinality of bases for the almost linear space X is unique. In the case $X = W_X + V_X$, we prove that $B' = \{x'_1, ..., x'_n\}$ is a basis for the algebraic dual $X^{\#}$ of X if $B = \{x_1, ..., x_n\}$ is a basis for the almost linear space X. And we have an example $X(\neq W_X + V_X)$ which has no such a basis.

1. Introduction

An *almost linear space* (als) is a set X together with two mappings $s : X \times X \to X$ and $m : \mathbb{R} \times X \to X$ satisfying the conditions $(L_1) - (L_8)$ given below. For $x, y \in X$ and $\lambda \in \mathbb{R}$ we denote $s(x,y)$ by $x + y$ and $m(\lambda, x)$ by λx, when these will not lead to misunderstandings. Let $x, y, z \in X$ and $\lambda, \mu \in \mathbb{R}$. (L_1) $x + (y + z) = (x + y) + z$; (L_2) $x + y = y + x$; (L_3) There exists an element $0 \in X$ such that $x + 0 = x$ for each $x \in X$; (L_4) $1x = x$; (L_5) $\lambda(x + y) = \lambda x + \lambda y$; (L_6) $0x = 0$; (L_7) $\lambda(\mu x) = (\lambda\mu)x$; (L_8) $(\lambda + \mu)x = \lambda x + \mu x$ for $\lambda \geq 0,\ \mu \geq 0$.

We denote $-1x$ by $-x$, if there is no confusion likely, and in the sequel $x - y$ means $x + (-y)$.

Note that $(\lambda + \mu)x = \lambda x + \mu x$ for every scalars $\lambda, \mu \in \mathbb{R}$ in linear space, and $x - x$ need not be equal to zero for every x in almost linear space.

If X is an als then we have: (1) The element 0 in (L_3) is unique. (2) $\lambda 0 = 0$ for each $\lambda \in \mathbb{R}$. (3) For each $x \in X$ and $\lambda \leq 0,\ \mu \leq 0,\ (\lambda + \mu)x = \lambda x + \mu x$. (4) If $x \in X$ is such that $x - x = 0$, then $(\lambda + \mu)x = \lambda x + \mu x$ for all $\lambda, \mu \in \mathbb{R}$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

A nonempty subset Y of an als X is called an *almost linear subspace* of X, if for each $y_1, y_2 \in Y$ and $\lambda \in \mathbb{R}$, $s(y_1, y_2) \in Y$ and $m(\lambda, y_1) \in Y$. An almost linear subspace Y of X is called a *linear subspace* of X if $s : Y \times Y \to Y$ and $m : \mathbb{R} \times Y \to Y$ satisfy all the axioms of a linear space.

For an als X we introduce the following two sets;

$$V_X = \{x \in X : x - x = 0\}, \tag{1.1}$$

$$W_X = \{x \in X : x = -x\}. \tag{1.2}$$

Then, we have the following properties: (1) The set V_X is a linear subspace of X, and it is the largest one. (2) The set W_X is an almost linear subspace of X and $W_X = \{x - x : x \in X\}$. (3) The als X is a linear space $\Longleftrightarrow V_X = X \Longleftrightarrow W_X = \{0\}$, and $V_X \cap W_X = \{0\}$.

All notions and notations used and not defined in this paper can be found in [2], [3], and [4].

2. Basis For The Almost Linear Space

A subset B of the als X is called a *basis* for X if for each $x \in X - \{0\}$ there exist unique sets $\{b_1, b_2, ..., b_n\} \subset B$, $\{\lambda_1, \lambda_2, ..., \lambda_n\} \subset \mathbb{R} - \{0\}$ (n depending on x) such that $x = \sum_{i=1}^{n} \lambda_i b_i$, where $\lambda_i > 0$ for $b_i \notin V_X$. Clearly, if B is a basis for X then $0 \notin B$.

In contrast to the case of a ls, there exists als which has no basis.

Examples 2.1. (1) Let $X = \{x \in R : x \geq 0\}$. Define $s(x, y) = max\{x, y\}$ and $m(\lambda, x) = x$ for $\lambda \neq 0$, $m(0, x) = 0$. The element $0 \in X$ is $0 \in R$. Then X is an als. We have $V_X = \{0\}$ and $W_X = X$. Furthermore, X has no basis[2].

(2) Let $X = \{[a, b] \subset R : a \leq b\}$. Define $s(A, B) = \{a + b : a \in A,\ b \in B\}$ and $m(\lambda, A) = \{\lambda a : a \in A\}$ for $A, B \in X$, $\lambda \in R$. Then X is an als. We have $V_X = \{\{a\} \in X : a \in R\}$ and $W_X = \{[-a, a] \in X : a \geq 0\}$. And $B = \{[-1, 1], \{1\}\}$ is a basis for X. Also, $Y = \{[a, b] \in X : a \leq 0,\ b \geq 0\}$ is an almost linear subspace of X. And $B_1 = \{[-1, 0], [0, 1]\}$ is a basis for Y.

Definition 2.2. Let $B = \{b_1, ..., b_n\}$ is a subset of the als X. If the equation

$$\lambda_1 b_1 + ... + \lambda_n b_n \ = \ \mu_1 b_1 + ... + \mu_n b_n \ (\lambda_i, \mu_i \geq 0 \ if \ b_i \notin V_X)$$

has the only solution

$$\lambda_1 = \mu_1, \ \lambda_2 = \mu_2, ..., \lambda_n = \mu_n,$$

then B is called an almost linearly independent set. If there are other solutions, then B is called an almost linearly dependent set.

Definition 2.3. If $B = \{b_1, ..., b_n\}$ is a subset of the als X and $X = \{\sum_{i=1}^{n} \lambda_i b_i | \lambda_i \in R, \lambda_i \geq 0 \ if \ b_i \notin V_X\}$, then we say that B almost span X and $\sum_{i=1}^{n} \lambda_i b_i$ is an almost linear combination of $b_1, b_2, ..., b_n$.

Proposition 2.4. Let $B = \{b_1, ..., b_n\}$ is a subset of the als X. Then B is a basis for X if and only B is an almost linearly independent set and B almost span X.

Proof. Assume that B is an almost linearly independent set and B almost span X. Given $x \in X - \{0\}$. Since B almost span X, there exist $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}$ where $\alpha_i \geq 0$ if $b_i \notin V_X$ such that $x = \alpha_1 b_1 + \alpha_2 b_2 + ... + \alpha_n b_n$. Suppose x has another representation, by, $x = \beta_1 b_1 + \beta_2 b_2 + ... + \beta_n b_n$, where $\beta_i \geq 0$ if $b_i \notin V_X$. Then

$$\alpha_1 b_1 + \alpha_2 b_2 + ... + \alpha_n b_n = \beta_1 b_1 + \beta_2 b_2 + ... + \beta_n b_n.$$

Since B is an almost linearly independent set, $\alpha_i = \beta_i$, $i = 1, 2, ..., n$. Therefore B is a basis for X.

To prove the converse, let B be a basis for the als X. Clearly, B almost span X. We must show that B is an almost linearly independent set. It is sufficient to show that if $\lambda_1 b_1 + ... + \lambda_n b_n = 0$ then $\lambda_1 = ... = \lambda_n = 0$. Indeed, suppose $\lambda_k \neq 0$, then $\lambda_1 b_1 + ... + 2\lambda_k b_k + ... + \lambda_n b_n = \lambda_k b_k$. This contradicts, since B is a basis and $\lambda_k b_k \neq 0$. Thus $\lambda_k = 0$. Therefore B is an almost linearly independent set.

Theorem 2.5[2]. Let B be a basis of the als X. Then there exists a basis B' of X with the property that for each $b' \in B' - V_X$ we have $-b' \in B' - V_X$. Moreover $card(B - V_X) = card(B' - V_X)$.

Proposition 2.6[2]. If the als X has a basis then W_X has a basis.

Proposition 2.7. Let X be an als with a basis. Then

(1) The relations $x+y=x+z,\ x,y,z\in X$ imply that $y=z$,

(2) The relations $w_1+v_1=w_2+v_2,\ w_i\in W_X,\ v_i\in V_X,\ i=1,2$ imply that $w_1=w_2$ and $v_1=v_2$.

Proof. (1) Let B be a basis for the als X and let $x=\sum_{i=1}^{n}\alpha_i b_i$, $y=\sum_{i=1}^{n}\beta_i b_i$, $z=\sum_{i=1}^{n}\gamma_i b_i$ where $b_i\in B$ and $\alpha_i,\beta_i,\gamma_i\geq 0$ if $b_i\notin V_X$, $i=1,2,...,n$. Then

$$x+y=\sum_{i=1}^{n}\alpha_i b_i+\sum_{i=1}^{n}\beta_i b_i=\sum_{i=1}^{n}(\alpha_i+\beta_i)b_i,$$

$$x+z=\sum_{i=1}^{n}\alpha_i b_i+\sum_{i=1}^{n}\gamma_i b_i=\sum_{i=1}^{n}(\alpha_i+\gamma_i)b_i.$$

Thus $x+y=x+z$ implies $\alpha_i+\beta_i=\alpha_i+\gamma_i$ $i=1,2,...,n$ since B is a basis. Hence $y=z$ if $x+y=x+z$.

(2) Let $w_1+v_1=w_2+v_2$, where $w_1,w_2\in W_X$, $v_1,v_2\in V_X$. Then $w_1=w_1+(v_1-v_1)=(w_1+v_1)-v_1=(w_2+v_2)-v_1=w_2+v_2-v_1$. Also, $w_1=-w_1=-w_2-v_2+v_1=w_2-v_2+v_1$. Hence $2w_1=2w_2$, so $w_1=w_2$. And $v_1=v_2$ by (1).

In Examples 2.1(2), $X=W_X+V_X$ and $B=\{[-1,1],\{1\}\}$ is a basis for the als $X=W_X+V_X$. Furthermore, $\{[-1,1]\}$ is a basis for W_X and $\{\{1\}\}$ is a basis for V_X. In general, we have the following result.

Proposition 2.8. Let X be an als with basis and $X=W_X+V_X$. Then we can choose a basis $B=B_1\cup B_2$ for X, where B_1 is a basis for W_X and B_2 is a basis for V_X.

Proof. If X has a basis then W_X has a basis by Proposition 2.6. Let B_1 be a basis for W_X and B_2 a basis for the linear space V_X. By Proposition 2.7(2), $B=B_1\cup B_2$ is a basis for W_X+V_X.

Lemma 2.9. If $B=\{b_1,b_2,...,b_n\}$ is a basis for the almost linear space X, then every set with more than n elements in X is an almost linearly dependent.

Proof. Let $B' = \{v_1, v_2, ..., v_m\}$ be any set of m elements in X, where $m > n$. Since B is a basis, each v_i can be expressed as an almost linear combination of the elements in B, say,

$$v_i = \lambda_{1i}b_1 + \lambda_{2i}b_2 + ... + \lambda_{ni}b_n \tag{2.1}$$

where $\lambda_{ji} \geq 0$ if $b_j \notin V_X$, $i = 1, 2, ..., m$. Consider the following equation

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_m v_m = \beta_1 v_1 + \beta_2 v_2 + ... + \beta_m v_m. \tag{2.2}$$

To show that B' is an almost linearly dependent, it is sufficient to show that there exists nonnegative nontrivial solution of (2.2).

We may assume that α_i, β_i are nonnegative, $i = 1, 2, ..., m$. Then we have

$$(\alpha_1\lambda_{j1} + \alpha_2\lambda_{j2} + ... + \alpha_m\lambda_{jm})b_j = \alpha_1\lambda_{j1}b_j + \alpha_2\lambda_{j2}b_j + ... + \alpha_m\lambda_{jm}b_j,$$

$$(\beta_1\lambda_{j1} + \beta_2\lambda_{j2} + ... + \beta_m\lambda_{jm})b_j = \beta_1\lambda_{j1}b_j + \beta_2\lambda_{j2}b_j + ... + \beta_m\lambda_{jm}b_j$$

since $\lambda_{ji} \geq 0$ if $b_j \notin V_X$, $i = 1, 2, ..., m$. Using the equations in (2.1), we can rewrite (2.2) as

$$\sum_{j=1}^{n}(\alpha_1\lambda_{j1} + \alpha_2\lambda_{j2} + ... + \alpha_m\lambda_{jm})b_j = \sum_{j=1}^{n}(\beta_1\lambda_{j1} + \beta_2\lambda_{j2} + ... + \beta_m\lambda_{jm})b_j.$$

Since B is a basis for the als X, we have

$$\alpha_1\lambda_{j1} + \alpha_2\lambda_{j2} + ... + \alpha_m\lambda_{jm} = \beta_1\lambda_{j1} + \beta_2\lambda_{j2} + ... + \beta_m\lambda_{jm} \tag{2.3}$$

where $j = 1, 2, ..., n$. We can rewrite (2.3) as

$$\lambda_{j1}x_1 + \lambda_{j2}x_2 + ... + \lambda_{jm}x_m = 0 \tag{2.4}$$

where $x_i = \alpha_i - \beta_i$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$. Since (2.4) has more unknowns than equations, there exists a nontrivial solution of (2.4). Hence $\alpha_k \neq \beta_k$ for some k. Therefore B' is an almost linearly dependent.

Theorem 2.10. If $B = \{b_1, b_2, ..., b_n\}$ and $B' = \{b'_1, b'_2, ..., b'_m\}$ are two bases for the als X, then $n = m$.

Proof. Since B is a basis and B' is an almost linearly independent set, Lemma 2.9 implies that $m \leq n$. Similarly, since B' is a basis and B is an almost linearly independent, we also have $n \leq m$. Therefore $m = n$.

3. Basis For The Algebraic Dual Space Of ALS

Let X be an als. A functional $f : X \to \mathbb{R}$ is called an *almost linear functional* if the conditions (3.1) – (3.3) are satisfied.

$$f(x + y) = f(x) + f(y) \quad (x, y \in X) \tag{3.1}$$

$$f(\lambda x) = \lambda f(x) \quad (\lambda \geq 0,\ x \in X) \tag{3.2}$$

$$f(w) \geq 0 \quad (w \in W_X) \tag{3.3}$$

The functional $f : X \to \mathbb{R}$ is called a *linear functional* on X if it satisfies (3.1), and (3.2) for each $\lambda \in \mathbb{R}$. Then (3.3) is also satisfied.

Let $X^{\#}$ be the set of all almost linear functionals defined on the als X. We define two operations $s : X^{\#} \times X^{\#} \to X^{\#}$ and $m : \mathbb{R} \times X^{\#} \to X^{\#}$ as follows:

$$s(f_1, f_2)(x) = f_1(x) + f_2(x), \qquad \text{for } f_1, f_2 \in X^{\#},$$

$$m(\lambda, f)(x) = f(\lambda x), \qquad \text{for } \lambda \in \mathbb{R},\ f \in X^{\#},$$

for all $x \in X$. Clearly, $s(f_1, f_2) \in X^{\#}$, $m(\lambda, f) \in X^{\#}$, and s, m satisfy $(L_1) - (L_8)$ with $0 \in X^{\#}$ being the functional which is 0 at each $x \in X$. Therefore $X^{\#}$ is an als. $X^{\#}$ is called the *algebraic dual space* of the als X.

We denote $s(f_1, f_2)$ by $f_1 + f_2$ and $m(\lambda, f)$ by $\lambda \circ f$.

Let f be an almost linear functional on als X. Then we have: (1) If $x \in V_X$ then $f(\lambda x) = \lambda f(x)$ for $\lambda \in \mathbb{R}$. (2) $f \in V_{X^{\#}} \iff f$ is linear on $X \iff -1 \circ f = -f \iff f|_{W_X} = 0$.

Let als X have a basis. Then, we can choose a basis B such that $-x \in B - V_X$ if $x \in B - V_X$ by Theorem 2.5. For each $x_i \in B$ we can define functional

$$x_i' : X \to R \tag{3.4}$$

by $x_i'(x) = \lambda_i$, for $x = \sum_{j=1}^{n} \lambda_j x_j \in X$ where $x_j \in B$ and $\lambda_j \geq 0$ *if* $x_j \notin V_X$.

Proposition 3.1. The functional x_i' defined by (3.4) is an almost linear functional on the als X. In particular, if $x_i \in V_X$ then $x_i' \in V_{X^\#}$, and if $x_i \in W_X$ then $x_i' \in W_{X^\#}$.

Proof. Clearly, $x_i'(x+y) = x_i'(x) + x_i'(y)$ and $x_i'(\lambda x) = \lambda x_i'(x)$ for $x, y \in X$, $\lambda \geq 0$.

We show that $x_i'(w) \geq 0$ for $w \in W_X$. For each $x_i \in B$, put $y_i = x_i$ if $x_i \in V_X$ and $y_i = -x_i$ if $x_i \notin V_X$. Given $\sum_{i=1}^{n} \lambda_i x_i \in X$ where $\lambda_i \geq 0$ *if* $x_i \notin V_X$, $-x = \sum_{i=1}^{n} -\lambda_i x_i = \sum_{i=1}^{n} \mu_i y_i$ where $\mu_i = \lambda_i$ if $x_i \notin V_X$ and $\mu_i = -\lambda_i$ if $x_i \in V_X$.

If $w \in W_X$ then $w = \sum_{j=1}^{k} \lambda_j x_j$ where $\lambda_j \geq 0$, $x_j \notin V_X$. Indeed, if $x = \sum_{i=1}^{n} \lambda_i x_i \in W_X$ then $\sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n} \mu_i y_i$ since $x = -x$. If $x_i \in V_X$ then $x_i = y_i$. So, $\lambda_i = \mu_i$ since the representation is unique. But $\mu_i = -\lambda_i$ since $x_i \in V_X$. Thus $\lambda_i = 0$ if $x_i \in V_X$. Hence $x_i'(w) \geq 0$ for $w \in W_X$. Therefore $x_i' \in X^\#$. Also,

$$(x_i' + (-1) \circ x_i')(\sum_{j=1}^{n} \lambda_j x_j) = x_i'(\sum_{j=1}^{n} \lambda_j x_j) + x_i'(\sum_{j=1}^{n} \mu_j y_j) = \lambda_i + \mu_i = \lambda_i - \lambda_i = 0$$

if $x_i \in V_X$.

Hence $x_i' \in V_{X^\#}$ if $x_i \in V_X$. And, if $x_i \in W_X$ then $x_i = -x_i = y_i$ and $\lambda_i = \mu_i$. So we have

$$-1 \circ x_i'(\sum_{j=1}^{n} \lambda_j x_j) = x_i'(\sum_{j=1}^{n} -\lambda_j x_j) = x_i'(\sum_{j=1}^{n} \mu_j y_j) = \mu_i = \lambda_i = x_i'(\sum_{j=1}^{n} \lambda_j x_j).$$

Hence $x_i' \in W_{X^\#}$ if $x_i \in W_X$.

Theorem 3.2. Let X be an als and $X = W_X + V_X$. If $B = \{x_1, ..., x_n\}$ is a basis for the als X then $B' = \{x'_1, ..., x'_n\}$ given by (3.4) is a basis for the algebraic dual $X^\#$ of X.

Proof. B' is an almost linearly independent set since

$$\sum_{i=1}^{n} \alpha_i \circ x'_i(x) = \sum_{i=1}^{n} \beta_i \circ x'_i(x) \quad (x \in X)$$

with $x = x_j$ gives

$$\alpha_j = \sum_{i=1}^{n} x'_i(\alpha_i x_j) = \sum_{i=1}^{n} \alpha_i \circ x'_i(x_j) = \sum_{i=1}^{n} \beta_i \circ x'_i(x_j) = \sum_{i=1}^{n} x'_i(\beta_i x_j) = \beta_j,$$

so that $\alpha_i = \beta_i,\ i = 1, 2, ..., n$.

We show that every $x' \in X^\#$ can be represented as an almost linear combination of the elements of B'. For given $x' \in X^\#$. Write $x'(x_i) = \alpha_i$ for each $x_i \in B$. If $x_i \notin V_X$ then $x_i \in W_X$, so $\alpha_i \geq 0$. Since x' is an almost linear functional on X

$$x'(x) = \sum_{j=1}^{n} \lambda_j \alpha_j$$

for every $x = \sum_{j=1}^{n} \lambda_j x_j \in X$ where $\lambda_j \geq 0\ if\ x_j \notin V_X$. On the other hand, by (3.4) we obtain

$$x'_j(x) = x'_j(\lambda_1 x_1 + ... + \lambda_n x_n) = \lambda_j.$$

Together,

$$\sum_{j=1}^{n} \alpha_j \circ x'_j(x) = \sum_{j=1}^{n} x'_j(\alpha_j x) = \sum_{j=1}^{n} \alpha_j x'_j(x) = \sum_{j=1}^{n} \alpha_j \lambda_j,$$

since $\alpha_j \geq 0$ if $x_j \notin V_X$. So, $x' = \sum_{j=1}^{n} \alpha_j \circ x'_j$. Hence B' almost span $X^\#$. Therefore B' is a basis for the als $X^\#$ by Proposition 2.4.

Remark. In Theorem 3.2, $X = W_X + V_X$ is essential. Indeed, in Examples 2.1(2) $Y = \{[a,b] : a \leq 0,\ b \geq 0\}$ is an als and $B = \{b_1 = [-1,0], b_2 = [0,1]\}$ is a basis for Y. Note that $W_Y = \{[-a,a] : a \geq 0\}$, $V_Y = \{\{0\}\}$ and $Y \neq W_Y + V_Y$. But $B' = \{b'_1, b'_2\}$ is not a basis for $Y^\#$. For example, the element $f = b'_1 - (-1) \circ b'_1 \in Y^\#$ cannot be written as an almost linear combination of b'_1, b'_2: Suppose $B' = \{b'_1, b'_2\}$ were a basis for $Y^\#$. Then $f = \alpha_1 \circ b'_1 + \alpha_2 \circ b'_2$ with both $\alpha'_i s$ non-negative. Now $(\alpha_1 \circ b'_1 + \alpha_2 \circ b'_2)(b_2) = \alpha_2 \geq 0$. However, $f(b_2) = -1$. Therefore, such $\alpha'_i s$ cannot exist.

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