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ON PROPERTIES OF COMPLEX ORDER FOR THE CLASSES OF UNIVALENT FUNCTIONS

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1. Introduction

Let $\mathcal{A}$ be the class of univalent functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1.1}$$

which are analytic in the unit disk $\Delta = \{z : |z| < 1\}$.

Let $S^*(\rho)$ be the subclass of $\mathcal{A}$ composing of functions which are starlike of order ρ. A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be starlike of order ρ $(\rho \neq 0)$ if and only if $z^{-1}f(z) \neq 0$ $(z \in \Delta)$ and

$$Re\left[1 + \frac{1}{\rho}\left\{\frac{zf'(z)}{f(z)} - 1\right\}\right] > 0 \qquad (z \in \Delta). \tag{1.2}$$

Let $K(\rho)$ be the subclass of $\mathcal{A}$ composing of functions which are convex of order ρ. A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be convex of order ρ $(\rho \neq 0)$ if and only if $f'(z) \neq 0$ $(z \in \Delta)$ and

$$Re\left[1 + \frac{1}{\rho}\left\{\frac{zf''(z)}{f'(z)}\right\}\right] > 0 \qquad (z \in \Delta). \tag{1.3}$$

Note that $f(z) \in K(\rho)$ if and only if $zf'(z) \in S^*(\rho)$.

Let $C(\rho)$ be the subclass of $\mathcal{A}$ composing of functions which are close-to-convex of order ρ. A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be close-to-convex of order ρ $(\rho \neq 0)$ if and only if there exists a function $g(z) \in K(1)$ such that $f(z) = zg'(z)$ and

$$Re\left[1 + \frac{1}{\rho}\left\{\frac{f'(z)}{g'(z)} - 1\right\}\right] > 0 \qquad (z \in \Delta). \tag{1.4}$$

Note that $C(1-\rho) = C(\rho)$, where $C(\rho)$ is the class of close-to-convex functions of order ρ $(0 \leq \rho < 1)$.

Let $\alpha S(\rho)$ be the subclass of $\mathcal{A}$ composing of functions which are α-spirallike of order ρ. Let $\mathcal{B}$ represent the class of normalized holomorphic functions of positive real part, i.e., $P \in \mathcal{B}$ if and only if P is holomorphic in Δ and $P(0) = 1$.

$$Re\{P(z)\} > 0 \qquad (z \in \Delta). \tag{1.5}$$

If $f \in \alpha S$, the introduction of appropriate normalizing factors enables us to write

$$\sec\alpha\left[e^{i\alpha}\{zf'(z)/f(z)\} - i\sin\alpha\right]_{z=0} = 1.$$

This leads to useful representation formulas for members of αS in terms of function in $\mathcal{B}$ such that

$$\frac{zf'(z)}{f(z)} = \frac{\cos\alpha \,.\, P(z) + i\sin\alpha}{\cos\alpha + i\sin\alpha} \qquad (z \in \Delta), \tag{1.6}$$

or equivalently

$$f(z) = z\exp\left\{\cos\alpha \,.\, e^{-i\alpha}\int_0^z \frac{P(t)-1}{t}dt\right\} \qquad (z \in \Delta). \tag{1.7}$$

A function $f(z)$ belonging to the subclass $\alpha S(\rho)$ is said to be α-spirallike of order ρ $(\rho \leq 1, |\alpha| < \frac{\pi}{2})$ if and only if f is defined by (1.6).

In this paper, we investigate the several interesting properties and coefficient inequalities of functions belonging to these subclasses $S^*(\rho)$, $K(\rho)$, $C(\rho)$ and $\alpha S(\rho)$ of the class of univalent functions.

2. Preliminaries

In this chapter, investigate the several interesting properties for functions belonging to the classes $S^*(\rho)$, $K(\rho)$, $C(\rho)$ and $\alpha S(\rho)$.

Lemma 2.1. [7] *Let $q(z)$ be the univalent in Δ, $\theta(w)$ and $\varphi(w)$ be analytic in the domain D containing $q(\Delta)$, and $\varphi(w) \neq 0$ for $w \in q(\Delta)$. Set*

$$Q(z) = zq'(z)\varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + Q(z).$$

Suppose that

(i) *$Q(z)$ is starlike in Δ with $Q(0) = 0$, $Q'(z) \neq 0$, and*
(ii) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{\frac{\theta'(q(z))}{\varphi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z \in \Delta).$

If $p(z)$ is analytic in Δ with $p(0) = q(0)$, $p(\Delta) \subset D$, and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)) = h(z), \tag{2.1}$$

then $p(z) < q(z)$, and $q(z)$ is the best dominant of the subordination (2.1).

Theorem 2.2. *If a function $f(z)$ is in the class $S^*(\rho)$, then*

$$\left\{\frac{f(z)}{z}\right\}^a \prec \frac{1}{(1-z)^{2a\rho}}, \tag{2.2}$$

where a is a complex number, $a \neq 0$, and either $|2a\rho+1| \leq 1$ or $|2a\rho-1| \leq 1$. The function $1/(1-z)^{2a\rho}$ is the best dominant of the differential subordination (2.2).

Proof. Letting $q(z) = 1/(1-z)^{2a\rho}$, $\theta(w) = 1$, and $\varphi(w) = 1/a\rho w$ in Lemma 2.1, we see that $Q(z) = 2z/(1-z)$ and $h(z) = (1+z)/(1-z)$.

Therefore, $Q(z)$ is starlike in Δ, $Q(0) = 0$, $Q'(0) = 2 \neq 0$, and

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{\frac{1}{1-z}\right\} > 0 \qquad (z \in \Delta).$$

Thus, the conditions (i) and (ii) of Lemma2.1 are satisfied. Also, we see that $q(z)$ is univalent in Δ by Royster [1]. We define the function $p(z)$ by $p(z) = (f(z)/z)^a$ for $f(z) \in S^*(\rho)$. Then $p(z)$ is analytic in Δ,

$$p(z) = 1 + p_1(z) + p_2(z)^2 + \cdots,$$

and $p(z) \neq 0$ for $0 < |z| < 1$. Since

$$\theta(p(z)) + zp'(z)\varphi(p(z)) = 1 + \frac{1}{a\rho}\frac{zp'(z)}{p(z)} = 1 + \frac{1}{\rho}\left\{\frac{zf'(z)}{f(z)} - 1\right\}, \tag{2.3}$$

$f(z) \in S^*(\rho)$ implies that

$$1 + \frac{1}{\rho}\left\{\frac{zf'(z)}{f(z)} - 1\right\} \prec \frac{1+z}{1-z} = h(z). \tag{2.4}$$

Consequently, with the help of Lemma 2.1, we observe that

$$1 + \frac{1}{\rho}\left\{\frac{zf'(z)}{f(z)} - 1\right\} \prec \frac{1+z}{1-z} \Longrightarrow \left\{\frac{f(z)}{z}\right\}^a \prec \frac{1}{(1-z)^{2a\rho}}. \tag{2.5}$$

This completes the proof of Theorem 2.2.

Lemma2.3. (Jack[8]) *Let $w(z)$ be regular in Δ and such that $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0, then we have $z_0 w'(z_0) = kw(z_0)$, where k is real and $k \geq 1$.*

Theorem 2.4. *If a function $f(z)$ belonging to $\mathcal{A}$ satisfies*

$$\left|\frac{f'(z)}{g'(z)} - 1\right|^{\alpha} \left|\frac{zf''(z)}{g'(z)} - \frac{zf'(z)g''(z)^2}{g'(z)^2}\right|^{\beta} < |\rho|^{\alpha+\beta} \qquad (z \in \Delta) \tag{2.6}$$

for some $\alpha \geq 0$, $\beta \geq 0$, and $g(z) \in K(1)$, then $f(z) \in C(\rho)$.

Proof. Defining the function $w(z)$ by

$$w(z) = \frac{1}{\rho}\left\{\frac{f'(z)}{g'(z)} - 1\right\} \tag{2.7}$$

for $f(z)$ belonging to $\mathcal{A}$ and $g(z)$ belonging to $K(1)$, we see that $w(z)$ is regular in Δ and $w(0) = 0$. Noting that

$$\rho z w'(z) = \frac{zf''(z)}{g'(z)} - \frac{zf'(z)g''(z)}{g'(z)^2}, \tag{2.8}$$

we know that (2.6) can be written as

$$|\rho w(z)|^{\alpha} |\rho z w'(z)|^{\beta} < |\rho|^{\alpha+\beta}. \tag{2.9}$$

Suppose that there exists a point $z_0 \in \Delta$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1. \tag{2.10}$$

Then Lemma 2.3. leads us to

$$\begin{aligned} |\rho w(z_0)|^{\alpha} |\rho z_0 w'(z_0)|^{\beta} &= |\rho|^{\alpha+\beta} k^{\beta} \qquad (k \geq 1) \\ &\geq |\rho|^{\alpha+\beta} \end{aligned} \tag{2.11}$$

which contradicts our condition (2.6). Therefore, we conclude that $|w(z)| < 1$ for all $z \in \Delta$, that is, that

$$\left|\frac{1}{\rho}\left\{\frac{f'(z)}{g'(z)} - 1\right\}\right| < 1 \qquad (z \in \Delta).$$

This implies that

$$Re\left[1 + \frac{1}{\rho}\left\{\frac{f'(z)}{g'(z)} - 1\right\}\right] > 0 \qquad (z \in \Delta),$$

which proves that $f(z) \in C(\rho)$.

Letting $g(z) = z \in K(1)$ in Theorem 2.4, we have

Corollary 2.5. *If a function $f(z)$ belonging to $\mathcal{A}$ satisfies*

$$|f'(z) - 1|^{\alpha}|zf''(z)|^{\beta} < |\rho|^{\alpha+\beta} \qquad (z \in \Delta) \tag{2.12}$$

for some $\alpha \geq 0$ and $\beta \geq 0$, then $f(z) \in C(\rho)$.

Proposition 2.6. *If*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \Delta), \tag{2.13}$$

$f \in \alpha S(\rho)$ and w is a function holomorphic in Δ, then

$$\begin{aligned}\left[\sum_{k=1}^{n-1}\{ke^{i\alpha}\sec\alpha + (1 - 2\rho - i\tan\alpha)\}a_k z^k\right] w(z) \\ = \sum_{k=2}^{n}\{ke^{i\alpha}\sec\alpha - (1 + i\tan\alpha)\}a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k\end{aligned} \tag{2.14}$$

Proof. If w is a function holomorphic in Δ, and

$$P(z) = \{1 + (1 - 2\rho)w(z)\}/\{1 - w(z)\} \qquad (z \in \Delta), \tag{2.15}$$

then $P \in \mathcal{B}$. Consequently, the representation given in (1.5) may be written

$$e^{i\alpha} \sec\alpha \, . \, \frac{zf'(z)}{f(z)} - i\tan\alpha = \frac{1 + (1 - 2\rho)w(z)}{1 - w(z)}. \tag{2.16}$$

Hence

$$\begin{aligned} &\left[\sum_{k=1}^{\infty}\{ke^{i\alpha}\sec\alpha + (1 - 2\rho - i\tan\alpha)\}a_k z^k\right] w(z) \\ &\qquad = \sum_{k=2}^{\infty}\{ke^{i\alpha}\sec\alpha - (1 + i\tan\alpha)\}a_k z^k, \end{aligned} \tag{2.17}$$

or

$$\begin{aligned} &\left[\sum_{k=1}^{n-1}\{ke^{i\alpha}\sec\alpha + (1 - 2\rho - i\tan\alpha)\}a_k z^k\right] w(z) \\ &\qquad = \sum_{k=2}^{n}\{ke^{i\alpha}\sec\alpha - (1 + i\tan\alpha)\}a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k. \end{aligned}$$

with the last sum convergent in Δ and $n = 2, 3, \cdots$.

3. Main Properties

In this chapter, we investigate the coefficient inequalities for functions belonging to classes $S^*(\rho)$, $K(\rho)$, $C(\rho)$ and $\alpha S(\rho)$.

Theorem 3.1. *If a function $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}(n-1+|\rho|)|a_n| \leq |\rho| \qquad (\rho \neq 0), \tag{3.1}$$

then $f(z) \in S^*(\rho)$.

Proof. From (1.2), since

$$Re\left[1+\frac{1}{\rho}\left\{\frac{zf'(z)}{f(z)}-1\right\}\right] > 0 \qquad (z \in \Delta) \Longleftrightarrow f(z) \in S^*(\rho),$$

we can transform

$$\left|\frac{1}{\rho}\left\{\frac{zf'(z)}{f(z)}-1\right\}\right| \leq 1$$

to $\sum_{n=2}^{\infty}(n-1+|\rho|)|a_n| \leq |\rho| \quad (\rho \neq 0) \Longrightarrow f(z) \in S^*(\rho)$. Therefore this completes the proof of Theorem 3.1.

Theorem 3.2. *If a function $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n(n-1+|\rho|)|a_n| \leq |\rho| \qquad (\rho \neq 0), \tag{3.2}$$

then $f(z) \in K(\rho)$.

Proof. From (1.3), since

$$Re\left[1+\frac{1}{\rho}\left\{\frac{zf''(z)}{f'(z)}\right\}\right] > 0 \qquad (z \in \Delta) \Longleftrightarrow f(z) \in K(\rho),$$

we can transform

$$\left|\frac{1}{\rho}\left\{\frac{zf''(z)}{f'(z)}\right\}\right| \leq 1$$

to $\sum_{n=2}^{\infty} n(n-1+|\rho|)|a_n| \leq |\rho| \quad (\rho \neq 0) \Longrightarrow f(z) \in K(\rho)$. Therefore this completes the proof of Theorem 3.2.

Theorem 3.3. *If a function $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n|a_n| \leq |\rho| \qquad (\rho \neq 0), \tag{3.3}$$

then $f(z) \in C(\rho)$.

Proof. From (1.4), since

$$Re\left[1 + \frac{1}{\rho}\left\{\frac{f'(z)}{g'(z)} - 1\right\}\right] > 0 \qquad (z \in \Delta) \Longleftrightarrow f(z) \in C(\rho),$$

$f(z) = zg'(z)$ and $g(z) \in K(1)$. We can transform

$$\left|\frac{1}{\rho}\left\{\frac{f'(z)}{g'(z)} - 1\right\}\right| \leq 1$$

to $\sum_{n=2}^{\infty} n|a_n| \leq |\rho| \qquad (\rho \neq 0) \Longrightarrow f(z) \in C(\rho)$. Therefore this completes the proof of Theorem 3.3.

Proposition 3.4. *If a function $f(z) \in \alpha S(\rho)$, then for larger n by induction argument establishing bounds are*

$$4(1-\rho)\sum_{k=1}^{n-1}(k-\rho)|a_k|^2 \geq (n-1)^2 \sec^2 \alpha \, . \, |a_n|^2. \tag{3.4}$$

Proof. In (2.14), let $z = re^{i\theta}$, $0 < r < 1$, $0 \le \theta < 2\pi$, then

$$\begin{aligned}
&\sum_{k=1}^{n-1} |ke^{i\alpha}\sec\alpha + (1-2\rho - i\tan\alpha)|^2 |a_k|^2 r^{2k} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left| \sum_{k=1}^{n-1} [ke^{i\alpha}\sec\alpha + (1-2\rho - i\tan\alpha)] a_k r^k e^{i\theta k} \right|^2 d\theta \\
&\ge \frac{1}{2\pi}\int_0^{2\pi} \left| \sum_{k=1}^{n-1} [ke^{i\alpha}\sec\alpha + (1-2\rho - i\tan\alpha)] a_k r^k e^{i\theta k} \right|^2 |w(re^{i\theta})|^2 d\theta \\
&\ge \frac{1}{2\pi}\int_0^{2\pi} \left| \sum_{k=2}^{n} [ke^{i\alpha}\sec\alpha - (1+i\tan\alpha)] a_k r^k e^{i\theta k} + \sum_{k=n+1}^{\infty} b_k r^k e^{i\theta k} \right|^2 d\theta \\
&\ge \sum_{k=2}^{n} |ke^{i\alpha}\sec\alpha - (1+i\tan\alpha)|^2 \cdot |a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |b_k|^2 r^{2k} \\
&\ge \sum_{k=2}^{n} |ke^{i\alpha}\sec\alpha - (1+i\tan\alpha)|^2 \cdot |a_k|^2 r^{2k}. =
\end{aligned}$$

Letting r close to 1 and rewriting the preceding inequality, we obtain

$$\sum_{k=1}^{n-1} \{ke^{i\alpha}\sec\alpha + (1-2\rho - i\tan\alpha)|^2 - |ke^{i\alpha}\sec\alpha - (1+i\tan\alpha)|^2\} |a_k|^2$$
$$\ge |ne^{i\alpha}\sec\alpha - (1+i\tan\alpha)|^2 \cdot |a_n|^2$$

some simplification reduces this to

$$4(1-\rho)\sum_{k=1}^{n-1}(k-\rho)|a_k|^2 \ge (n-1)^2 \sec^2\alpha \cdot |a_n|^2$$

which is established for larger n by an induction argument.

Letting $n = 2$ in proposition 3.4, we have

Corollary 3.5.

$$4(1-\rho)^2 \geq \sec^2\alpha \,.\, |a_2|^2 \tag{3.5}$$

or

$$|a_2| \leq 2(1-\rho)\cos\alpha$$

which is equivalent to (3.5).

Theorem 3.6. *If a function* $f(z) \in \alpha S(\rho)$ *and*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \Delta),$$

then for all admissible α *and* ρ *and for each* n, *these sharp bounds are*

$$|a_n| \leq \prod_{k=0}^{n-2} \frac{|2(1-\rho)\cos\alpha \,.\, e^{-i\alpha} + k|}{k+1}, \qquad n = 2, 3, \cdots, \tag{3.6}$$

Proof. Fix $n, n \geq 3$, and suppose that (3.4) holds for $k = 2, 3, \cdots, n-1$. Then

$$|a_n|^2 \leq \frac{4(1-\rho)\cos^2\alpha}{(n-1)^2}\left\{(1-\rho) + \sum_{k=2}^{n-1}(k-\rho)\prod_{j=0}^{k-2}\frac{|2(1-\rho)\cos\alpha \,.\, e^{-i\alpha}+j|^2}{(j+1)^2}\right\}$$

$$|a_n|^2 \leq \frac{4(1-\rho)\cos^2\alpha}{(n-1)^2}\left\{(1-\rho) + \sum_{k=2}^{n-2}(k-\rho)\prod_{j=0}^{k-2}\frac{|2(1-\rho)\cos\alpha \,.\, e^{-i\alpha}+j|^2}{(j+1)^2}\right.$$

$$\left. + (n-1-\rho)\prod_{j=0}^{n-3}\frac{|2(1-\rho)\cos\alpha \,.\, e^{-i\alpha}+j|^2}{(j+1)^2}\right\}$$

$$= \prod_{j=0}^{n-3}\frac{|2(1-\rho)\cos\alpha \,.\, e^{-i\alpha}+j|^2}{(j+1)^2} \,.\, \left\{\frac{(n-2)^2 + 4(n-1-\rho)(1-\rho)\cos^2\alpha}{(n-1)^2}\right\}$$

$$= \prod_{j=0}^{n-2}\frac{|2(1-\rho)\cos\alpha \,.\, e^{-i\alpha}+j|^2}{(j+1)^2} \tag{3.7}$$

Therefore

$$|a_n| \leq \prod_{k=0}^{n-2} \frac{|2(1-\rho)\cos\alpha \,.\, e^{-i\alpha} + k|}{k+1}.$$

This completes the proof of Theorem 3.6.

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