The Pure and Applied Mathematics **3** (1996), No 1, pp. 41–46
J. Korea Soc. of Math. Edu. (Series B)

SOME IDENTITIES INVOLVING FACTORIALS

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ABSTRACT. We obtain some interesting identities involving factorials by using the theory of Bessel functions.

1. Introduction

We find [1, p. 294; Theorem A.7.] an interesting identity: For n any positive integer,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\Gamma(\nu+k)}{\Gamma(\nu+k-m)}=0, \qquad m=0,1,\ldots,n-1, \tag{1}$$

where ν is an arbitrary real (or complex) number and Γ denotes the well-known Gamma function one of whose important properties is

$$\Gamma(n+1)=n!, \qquad n=0,1,2,\ldots. \tag{2}$$

Setting $\nu=\ell+1$ a positive integer greater than or equal to n in (1) with the aid of (2) leads to an interesting identity involving factorials:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\ell+k)!}{(\ell+k-m)!}=0, \qquad m=0,1,\ldots,n-1. \tag{3}$$

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 00A22; Secondary 33E99.
Key words and phrases . Gamma function, Bessel functions.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

In this note we are aiming at getting some identities involving factorials of similar character as in (3) by using the theory of Bessel functions. For this purpose we introduce the definition of Bessel functions. We define, for n not a negative integer,

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+n}}{2^{2k+n} k! \Gamma(1+n+k)}. \tag{4}$$

If n is a negative integer, we set

$$J_n(z) = (-1)^n J_{-n}(z). \tag{5}$$

For all finite z and n $J_n(z)$ is defined by the equations (4) and (5). Then the function $J_n(z)$ is called the Bessel function of the first kind of index n. Of course (4) is a solution of the Bessel's differential equation:

$$z^2 \omega'' + z\omega' + (z^2 - n^2)\omega = 0. \tag{6}$$

As noted in [2, p. 109], no other special functions have received such detailed treatment in readily available treatises as have the Bessel functions: The most striking example is Watson's exhaustive (804 pages) work; Watson [3].

2. Identities involving factorials

In [2, p. 120, Ex.1] we obtain the following identity:

$$\sum_{n=0}^{\infty} J_{2n+1}(x) = \frac{1}{2} \int_0^x J_0(y)dy. \tag{7}$$

Using the definition (4) and expanding the left hand side of (7) in powers of x, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} J_{2n+1}(x) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2n+1}}{2^{2k+2n+1} k!\Gamma(2+2n+k)} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k x^{2n+1}}{2^{2n+1} k!\Gamma(2+2n-k)} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} \frac{(-1)^k}{k!\Gamma(2+2n-k)}\right\}\left(\frac{x}{2}\right)^{2n+1}\end{aligned} \tag{8}$$

On the other hand, we get

$$\begin{aligned}\frac{1}{2}\int_0^x J_0(y)dy &= \frac{1}{2}\int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{2^{2n} n!\Gamma(1+n)} dy \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!n!}\int_0^x y^{2n} dy \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!n!}\left(\frac{x}{2}\right)^{2n+1}\end{aligned} \tag{9}$$

In view of (7) with the help of (8) and (9), equating the coefficients of $(x/2)^{2n+1}$, we obtain the following interesting identity:

Theorem 1. *Let n be a nonnegative integer. Then we have*

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!(2n+1-k)!} = \frac{(-1)^n}{(2n+1)n!n!}, \tag{10}$$

or equivalently,

$$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} = (-1)^n \binom{2n}{n}. \tag{11}$$

In [2, p. 122] we obtain Neumann's expansion:

$$\left(\frac{1}{2}z\right)^n = \sum_{k=0}^{\infty} \frac{(n+2k)(n+k-1)!J_{n+2k}(z)}{k!}, \qquad n \geq 1. \tag{12}$$

Multiplying both sides of (12) by $(z/2)^{-n}$ and using (4), we get

$$\begin{aligned} 1 &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{(-1)^\ell (n+2k)(n+k-1)! z^{2\ell+2k}}{k!\ell!\Gamma(1+n+2k+\ell)2^{2k+2\ell}} \\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} \frac{(-1)^\ell (n+2k-2\ell)(n+k-\ell-1)!}{(k-\ell)!\ell!(n+2k-\ell)!} \left(\frac{z}{2}\right)^{2k}, \end{aligned}$$

from which we also obtain an interesting identity:

Theorem 2. *Let n and k be positive integers. Then we have*

$$\sum_{\ell=0}^{k} \frac{(-1)^\ell (n+2k-2\ell)(n+k-\ell-1)!}{(k-\ell)!\ell!(n+2k-\ell)!} = 0, \tag{13}$$

or, equivalently,

$$\sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \frac{(n+2k-2\ell)(n+k-\ell-1)!}{(n+2k-\ell)!} = 0, \tag{14}$$

from which we observe that (14) *is an interesting companion of* (3).

In [2, p. 120] we find

$$\sin(z \sin\theta) = 2\sum_{n=0}^{\infty} J_{2n+1}(z)\sin[(2n+1)\theta]. \tag{15}$$

Differentiating each side of (15) with respect to θ and setting $\theta = 0$ in the resulting equation yields

$$z = 2\sum_{n=0}^{\infty}(2n+1)J_{2n+1}(z). \tag{16}$$

Applying (4) to (16), we have

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(2n+1)}{k!\Gamma(2+2n+k)}\left(\frac{z}{2}\right)^{2k+2n} \\ &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\frac{(-1)^k(2n+1-2k)}{k!\Gamma(2+2n-k)}\right\}\left(\frac{z}{2}\right)^{2n}, \end{aligned}$$

from which we obtain the following identity:

Theorem 3. *Let n be a positive integer. Then we have*

$$\sum_{k=0}^{n}\frac{(-1)^k(2n+1-2k)}{k!(2n+1-k)!} = 0, \tag{17}$$

or, equivalently,

$$\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}(2n+1-2k) = 0. \tag{18}$$

Setting $\theta = \frac{\pi}{2}$ in (15), we obtain

$$\sin z = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(z). \tag{19}$$

Applying (4) to (19) and using the Maclaurin series of $\sin z$, we have

$$\begin{aligned} \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n+k}z^{2k+2n+1}}{2^{2k+2n}k!\Gamma(2+2n+k)} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^n}{k!\Gamma(2+2n-k)}\frac{z^{2n+1}}{2^{2n}}, \end{aligned}$$

from which we obtain the following identity:

Theorem 4. *Let n be a nonnegative integer. Then we have*

$$\sum_{k=0}^{n} \binom{2n+1}{k} \frac{1}{2^{2n}} = 1. \tag{20}$$

We conclude this note by posing a natural question: Show that identities (10), (13), (17) and (20) are true without using the theory of Bessel functions.

References

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