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ON THE SOLUTIONS OF THREE ORDER DIFFERENTIAL EQUATION WITH NON-NEGATIVE COEFFICIENTS

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1. Introduction

We consider the third order linear homogeneous differential equation

$$L_3(y) = y''' + P(x)y' + Q(x)y = 0 \tag{E}$$

$P(x) \geq 0$, $Q(x) > 0$ and $P(x)/Q(x)$ is nondecreasing on $[a, \infty)$ for some real number a. (1)

In this paper we discuss the distribution of zeros of solutions and a condition of oscillatory for equation (E).

(E) is said to be disconjugate on $[a, \infty)$ if no nontrivial solution of (E) has more than two zeros on $[a, \infty)$.

A nontrivial solution of (E) is said to be oscillatory on $[a, \infty)$, if it has an infinite number of zeros on $[a, \infty)$. The nontrivial solution of (E) is non-oscillatory if it is not oscillatory.

If (E) has an oscillatory solution, it is said to be oscillatory. And if all solutions of (E) are non-oscillatory then (E) is said to be non-oscillatory.

We give some basic definitions.

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2. Preliminaries

Definition 2.1. $L_3^*(z) = (z'' + P(x)z)' - Q(x)z = 0$ is adjoint of (E). (E^*)

Definition 2.2. $c \in [a, \infty)$ and $U_i(x, c), i = 1, 2$ be pair of solutions determined by the initial conditions at $x = c$.

(a) $U_1(x, c)$; $y(c) = 0,\ y'(c) = 1,\ y''(c) = 0$; first principal solution.
(b) $U_2(x, c)$; $y(c) = 0,\ y'(c) = 0,\ y''(c) = 1$; second principal solution.

Definition 2.3. Let $D_2(y) = y'' + P(x)y$, be second order differential operator and $c \in [a, \infty)$.

(a) $U_1^*(x, c)$; $Z(c) = 0,\ Z'(c) = 1,\ D_2Z(c) = 0$; first principal solution of (E^*) at $x = c$.
(b) $U_2^*(x, c)$; $Z(c) = 0,\ Z'(c) = 0,\ D_2Z(c) = 1$; second principal solution of (E^*) at $x = c$.

The wronskian of any two solutions of (E) is a solution of (E^*) and converse holds. Thus,

$$U_2^*(x, c) = W(U_1, U_2) = U_1U_2' - U_2U_1'$$
$$U_2(x, c) = W(U_1^*, U_2^*) = U_1^*U_2^{*\prime} - U_2^*U_1^{*\prime}$$

Differentiating these identities yields followings.

$$\begin{aligned}
U_2^{*\prime}(x, c) &= U_1U_2'' - U_2U_1'' \\
D_2U_2^*(x, c) &= U_1'U_2'' - U_2'U_1'' \\
U_2'(x, c) &= U_1^*D_2U_2^* - U_2^*D_2U_1^* \\
U_2''(x, c) &= U_1^{*\prime}D_2U_2^* - U_2^{*\prime}D_2U_1^* \\
D_2U_2(x, c) &= U_1^{*\prime}U_2^{*\prime\prime} - U_2^{*\prime}U_1^{*\prime\prime}
\end{aligned} \tag{2}$$

3. Main Theorem

Lemma 3.1. *Let (E) be disconjugate on $[a, \infty)$ and let its coefficients satisfy (1). If $U_2''(x,a)$ has a zero on (a, ∞) with $x = t$, being the first zero of $U_2''(x,a)$ then*

(a) *$U_2''(x,a)$ has a second zero $t_2 \in (t_1, \infty)$.*

(b) *$U_2'(x,a)$ has exactly one zero $s_1 \in (t_1, t_2)$ and $U_2'(x,a) < 0$ on (s_1, ∞).*

Proof. Assume $U_2''(x,a)$ has a zero at $x = t_1$.

Suppose $U_2'(x,a) > 0$ on (a, ∞). Then $U_2'''(x,a) < 0$ which implies $U_2''(x,a)$ is decreasing on (a, ∞). Therefore, $U_2''(x,a) < 0$ on (t_1, ∞). Let the first such zero of $U_2'(x,a)$ be s_1 and assume $U_2'(x,a)$ has a second zero s_2. Then $U_2'(x,a) < 0$ on (s_1, s_2).

From the identity $U_2^*(x,a) = W(U_1, U_2)$, we find $U'(s_2, a) < 0$. Let

$$\lambda_1(x) = \frac{U_1'(x,a)}{U_2'(x,a)}.$$

We find $\lambda_1(x) \to \infty$ as $x \to s_2$ on (s_1, s_2).

$$\text{And} \quad \lambda_1'(x) = \frac{U_2'(x,a)U_1''(x,a) - U_2''(x,a)U_1'(x,a)}{(U_2'(x,a))^2} = -\frac{D_2U_2^*(x,a)}{(U_2'(x,a))^2}, \text{ on } (s_1, s_2).$$

since $D_2U_2^*(x,a) = 1 + \int_a^x Q(t)U_2^*(t,a)dt > 0$, $\lambda_1'(x) < 0$ on (s_1, s_2) and this contradicts $\lambda_1(x) \to \infty$. Therefore, $U_2'(x,a)$ has exactly one zero $s_1 \in (t_1, \infty)$. If $U_2''(x,a)$ does not have a zero on $(t_1 \infty)$, then $U_2''(x,a) < 0$ and $U_2'(x,a) < 0$ on same interval and we conclude that $U_2(x,a)$ has a zero, contradicting the fact $U_2(x,a) > 0$ on (a, ∞). thus $U_2''(x,a)$ has a second zero $t_2 \in (t_1, \infty)$ and the Lemma follows.

Lemma 3.2. *Let (E) be disconjugate on $[a, \infty)$ and let its coefficients satisfy (1). Then $P(x)D_2U_2^*(x,a) + Q(X)U_2^*(x,a) > 0$ on (a, ∞).*

Proof. Since $U_2^*(x,a)$ is a solution of (E^*), we have $[U_2^{*\prime\prime}(x,a) + P(x)U_2^*(x,a)]' = Q(x)U_2^*(x,a)$.

Integrating from a to x,

$$U_2^{*\prime\prime}(x,a) + P(x)U_2^*(x,a) = 1 + \int_a^x Q(t)U_2^*(t,a)dt. \text{ Therefore}$$

$$\begin{aligned} U_2^{*\prime}(x,a) &= (x-a) + \int_a^x \int_a^t Q(s)U_2^*(s,a)ds dt - \int_a^x P(t)U_2^*(t,a)dt \\ &= (x-a) + \int_a^x (x-t)Q(t)U_2^*(t,a)dt - \int_a^x P(t)U_2^*(t,a)dt. \end{aligned}$$

$$\begin{aligned} P(x)D_2U_2^*(x,a) + Q(x)U_2^{*\prime}(x,a) &= P(x) + P(x)\int_a^x Q(t)U_2^*(t,a)dt \\ + Q(x)(x-a) + Q(x)\int_a^x (x-t)Q(t)U_2^*(t,a)dt &- Q(x)\int_a^x P(t)U_2^*(t,a)dt \\ = P(x) + Q(x)(x-a) &+ Q(x)\int_a^x (x-t)Q(t)U_2^*(t,a)dt \\ &+ \int_a^x [P(x)Q(t) - Q(x)P(t)]U_2^*(t,a)dt. \end{aligned}$$

Since $P(x)/Q(x)$ is nondecreasing and $U_2^*(x,a) > 0$, it follows that $P(x)D_2U_2^*(x,a) + Q(x)U_2^{*\prime}(x,a) > 0$ on (a, ∞).

Theorem 3.1. *Let (E) be disconjugate on $[a, \infty)$ and let its coefficients satisfy (1). Assume $U_2''(x,a)$ has a zero at t_1. Then $U_2''(x,a)$ has a second zero at t_2, and $U_2''(x,a) > 0$ on (t_2, ∞), $a < t_1 < t_2$.*

Proof. Suppose $U_2''(x,a)$ has a zero on (t_2, ∞). Let t_3 be the first zero of $U_2''(x,a)$ on this interval. Then the identity $D_2U_2^*(x,a)$ implies that $U_1''(t_3,a) > 0$.

Let $\lambda_2(x) = \dfrac{U_1''(x,a)}{U_2''(x,a)}$. Then $\lambda_2(x) \to \infty$ as $t \to t_3$ on (t_1, t_3).

$$\lambda_2'(x) = \frac{U_1'''(x,a)U_2''(x,a) - U_2'''(x,a)U_1''(x,a)}{(U_2''(x,a))^2}.$$

Since $U_1(x,a)$, $U_2(x,a)$ are solution of (E) and from the identity of (2), we have

$$\lambda_2'(x) = -\frac{P(x)D_2U_2^*(x,a) + Q(x)U_2^{*\prime}(x,a)}{(U_2''(x,a))^2}.$$

By Lemma 3.2, the numerator is positive. Thus $\lambda_2'(x) < 0$ on (t_2, t_3). This is a contradiction and Lemma 3.3 follows.

In next, we give a criterion for the oscillation of (E)

Lemma 3.3 [4]. *If $2Q(x) - P'(x) \leq 0$ and not identically zero in any interval then (E) has a solution $U(x)$ for which*

$$\begin{aligned} F[U(x)] &= U'(x)^2 - 2U(x)U''(x) - P(x)U^2(x) \\ &= F[U(a)] + \int_a^x (2Q(t) - P'(t))U^2(t)dt \end{aligned}$$

is always negative. Consequently $U(x)$ is nonoscillatory.

Definition 3.1. If (E) has a non-trivial solution with three zeros on $[t, \infty)$, $t \in [a, \infty)$, then the first conjugate point $\eta_1(t)$ of $x = t$ is defined by $\eta_1(t) = inf\{x_3;\ t \leq x_1 \leq x_2 \leq x_3,\ y(x_i) = 0,\ i = 1,2,3,\ y \neq 0,\ L_3(y) = 0\}$

Lemma 3.4 [2]. *If (E) is non-oscillatory then either*

(i) *for each $t \in [a, \infty)$, (E) has $\eta_1(t) < \infty$ or*
(ii) *(E^*) is oscillatory.*

Theorem 3.2. *Let the coefficients of (E) satisfy* $P(x) \geq 0,\ Q(x) \geq 0$ *and* $P(x) + Q(x) \neq 0$ *on* $[a,\ \infty)$. *If* $\eta_1(t) < \infty$ *for each* $t \in [a,\ \infty)$ *and* $2Q(x) - P'(x) \leq 0$, *then (E) is oscillatory.*

Proof. Assume (E) is non-oscillatory. By Lemma 3.4, (E^*) is oscillatory. since $P'(x) - Q(x) \geq P'(x) - 2Q(x) \geq 0$ and $2(P'(x) - Q(x)) - P'(x) = P'(x) - 2Q(x) \geq 0$, a result of Lemma 3.3 implies (E^*) has a non-oscillatory solution. this is a contradiction and (E) is oscillatory.

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