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NOTES ON A SYMMETRIC BILINEAR FORM ASSOCIATED WITH REGULAR DIRICHLET FORM

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ABSTRACT. We will show how bilinear form $\mathcal{E}_\mu$ related with some smooth measures can be extended to the $L^2(\mathbb{R}^n, \mathbb{C})$ setting.

1. Introduction

We consider a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ where X is locally compact separable metric space and m is a positive Radon measure on X with $Supp[m] = X$. S is the family of all smooth measures on X. Let $M = (\Omega, X_t, \zeta, P_x)$ be a Hunt process on X which is m-symmetric and associated with $(\mathcal{E}, \mathcal{F})$. For a given smooth measure μ, we denote by A_μ the unique positive continuous additive functional such that μ is the Revuz measure of A_μ. Let $\mu = \mu_+ - \mu_-$ be a signed Borel measure on X. If μ_+ and μ_- are smooth measures, then we write $\mu \in S - S$. For a Borel measure v on X, $L^2(X, v)$ is sometimes written $L^2(v)$ when the underlying context is clear.

For $\mu \in S - S$, we put

$$\mathcal{E}_\mu(f, g) = \mathcal{E}(f, g) + \int_X f(x)g(x)\mu(dx)$$

for all $f, g \in \mathcal{F} \cap L^2(|\mu| + m)$. We consider the case where X is the Euclidean space $\mathbb{R}^n$ and m is a Lebesgue measure on $\mathbb{R}^n$. It is essential to quantum mechanics

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX that functions are from the space $L^2(\mathbb{R}^n, \mathbb{C})$ of square-integrable (with respect to Lebesgue measure), complex-valued functions.

In this paper we extend $\mathcal{E}_\mu$ to $L^2(\mathbb{R}^n, \mathbb{C})$ setting and find self-adjoint operator which represent the extension of $\mathcal{E}_\mu$.

2. Extension of $\mathcal{E}_\mu$ to $L^2(\mathbb{R}^n, \mathbb{C})$

Let us use the short notation $L^2(\mu)$ for $L^2(X, \mu)$, for $\mu \in S - S$, we put

$$\mathcal{E}_\mu(u,v) = \mathcal{E}(u,v) + \int_X u(x)v(x)\mu(dx)$$

for all $u, v \in \mathcal{F} \cap L^2(|\mu| + m)$

Theorem 1. *If $\mathcal{E}_\mu$ is bounded below, densely defined and closed, then there exist a unique, densely defined self-adjoint operator H^μ which is bounded below and satisfies $(H^\mu u, v) = \mathcal{E}_\mu(u,v)$ for all $u \in D(H^\mu)$ and $v \in D(\mathcal{E}_\mu)$*

Proof. See[4] Theorem 2.6

For $\alpha \geq 0$, μ and ν in $S - S$,$f \subset \mathcal{B}(X)$, we set

$$U_\nu^{\alpha+\mu} f(x) = E_x[\int_0^\infty e^{-\alpha t - A_t^\mu} f(X_t) dA_t^\nu]$$

provided the right hand side makes sense. When $\nu = m$, we simply write $U^{\alpha+\mu} f$ for $U_\nu^{\alpha+\mu} f$.

In the following Theorem 2, we consider the case where X is the Euclidean space $\mathbb{R}^n$ and m is a Lebesgue measure on $\mathbb{R}^n$. If ψ is a function in $L^2(\mathbb{R}^n, \mathbb{C})$ (space of square integrable, complex valued functions), we denote by ψ_1 its real part and by ψ_2 its imaginary part; i.e., $\psi = \psi_1 + i\psi_2$

Theorem 2. *Let $\mu \in S - S$ be such that*

$$U^{\alpha+\mu}(L^2(m)) \subset L^2(m)$$

for some $\alpha > 0$. Suppose that $\mathcal{E}_\mu$ is closed. If we define $\mathcal{E}_\mu^C$ by

$$\mathcal{E}_\mu^C(\psi,\varphi) = \mathcal{E}_\mu(\psi_1,\varphi_1) + \mathcal{E}_\mu(\psi_2,\varphi_2) + i[\mathcal{E}_\mu(\psi_2,\varphi_1) - \mathcal{E}_\mu(\psi_1,\varphi_2)]$$

for all $\psi, \varphi \in D(\mathcal{E}_\mu) + iD(\mathcal{E}_\mu) \subset L^2(\mathbb{R}^n, \mathbb{C})$,then $\mathcal{E}_\mu^C$ is densely defined, bounded below and closed.

Proof. Since $D(\mathcal{E}_\mu)$ is dense in $L^2(\mathbb{R}^n)$, $D(\mathcal{E}_\mu)+iD(\mathcal{E}_\mu)$ is dense in $L^2(\mathbb{R}^n, \mathbb{C})$. Since $U^{\alpha+\mu}(L^2(m)) \subset L^2(m), \mathcal{E}_\mu$ is bounded below [1. *Theorem*4.1].

Let A be some real number satisfying $\mathcal{E}_\mu(u,u) \geq A\|u\|^2$ for all $u \in D(\mathcal{E}_\mu)$ and let $\psi = \psi_1 + i\psi_2 \in D(\mathcal{E}_\mu) + iD(\mathcal{E}_\mu)$. Then we have $\mathcal{E}_\mu^C(\psi,\psi) \geq A[\|\psi_1\|^2 + \|\psi_2\|^2] = A\|\psi\|^2$ By the symmetry of $\mathcal{E}_\mu$,

To verify $\mathcal{E}_\mu^C$ is closed, it suffices to show that $D(\mathcal{E}_\mu^C)$ is complete under the norm

$$|||\psi|||^2 = \mathcal{E}_\mu^C(\psi,\psi) + (-A+1)\|\psi\|^2$$

Let (ψ_n) be a sequence in $D(\mathcal{E}_\mu^C)$ such that $|||\psi_n - \psi_m||| \to 0$ as $n, m \to \infty$. Then $\psi_n = \psi_{n,1} + i\psi_{n,2}$ for each $n \in N$, where $\psi_{n,1}, \psi_{n,2}$ are in $D(\mathcal{E}_\mu)$. By the symmetry of $\mathcal{E}_\mu$,

$$|||\psi_n - \psi_m|||^2 = \mathcal{E}_\mu^C(\psi_n - \psi_m, \psi_n - \psi_m) + (-A+1)\|\psi_n - \psi_m\|^2$$

$$= \mathcal{E}_\mu(\psi_{n,1} - \psi_{m,1}, \psi_{n,1} - \psi_{m,1}) + \mathcal{E}_\mu(\psi_{n,2} - \psi_{m,2}, \psi_{n,2} - \psi_{m,2})$$

$$+(-A+1)\|\psi_{n,1} - \psi_{m,1}\|^2 + (-A+1)\|\psi_{n,2} - \psi_{m,2}\|^2$$

$$= \mathcal{E}_\mu(\psi_{n,1} - \psi_{m,1}, \psi_{n,1} - \psi_{m,1}) + (-A+1)\|\psi_{n,1} - \psi_{m,1}\|^2$$

$$+\mathcal{E}_\mu(\psi_{n,2} - \psi_{m,2}, \psi_{n,2} - \psi_{m,2}) + (-A+1)\|\psi_{n,2} - \psi_{m,2}\|^2$$

$$= |||\psi_{n,1} - \psi_{m,1}|||^2 + |||\psi_{n,2} - \psi_{m,2}|||^2.$$

Since $|||\psi_n - \psi_m||| \to 0$, $|||\psi_{n,1} - \psi_{m,1}||| \to 0$ and $|||\psi_{n,2} - \psi_{m,2}||| \to 0$. Since $\mathcal{E}_\mu$ is closed, there exist ψ_1, ψ_2 in $D(\mathcal{E}_\mu)$ such that $|||\psi_{n,1} - \psi_1||| \to 0$ and $|||\psi_{n,2} - \psi_2||| \to 0$ as $n \to \infty$.

This means that $|||\psi_n - \psi||| \to 0$ as $n \to \infty$. And since $\psi = \psi_1 + i\psi_2 \in D(\mathcal{E}_\mu^C)$ we conclude that $\mathcal{E}_\mu^C$ is closed.

Let H^μ be a self-adjoint operator as in Theorem 1. If we define H_C^μ on $D(H^\mu) + iD(H^\mu)$ by $H_C^\mu(\psi_1 + i\psi_2) = H^\mu\psi_1 + iH^\mu\psi_2$, then H_C^μ is a self-adjoint operator on $D(H^\mu) + iD(H^\mu) \subset D(\mathcal{E}_\mu^C)$.

Theorem 3. *Under the conditions of Theorem 2,*

$$(H_C^\mu\psi, \varphi) = \mathcal{E}_\mu^C(\psi, \varphi)$$

for all $\psi \in D(H_C^\mu)$ *and* $\varphi \in (\mathcal{E}_\mu^C)$.

Proof. By Theorem 1, there exist a unique densely defined self-adjoint operator H^* which is bounded below and satisfies $(H^*\psi, \varphi) = \mathcal{E}_\mu^C(\psi, \varphi)$ for all $\psi \in D(H^*)$ and for all$\varphi \in (\mathcal{E}_\mu^C)$. From the linearity of H_C^μ, $(H_C^\mu\psi, \varphi) = \mathcal{E}_\mu^C(\psi, \varphi)$ for $\psi = \psi_1 + i\psi_2 \in D(H_C^\mu)$ and $\varphi = \varphi_1 + i\varphi_2 \in D(\mathcal{E}_\mu^C)$.

Using consequences [[4], Corollary 2.4 and Theorem 2.6, p.323] of the first representation Theorem and the simple fact (see e.q. [[5], p.279]) that two self-adjoint operators, one of which extends the other, are actually equal, one has $H^* = H_C^\mu$.

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