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ON THE PROPER QUADRATIC FIRST INTEGRALS IN SYMPLECTIC MANIFOLDS

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1. Introduction

Classical mechanics begins with some variants of Newton's laws. Lagrangian mechanics describes motion of a mechanical system in the configuration space which is a differential manifold defined by holonomic constraints. For a conservative system, the equations of motion are derived from the Lagrangian function on Hamilton's variational principle as a system of the second order differential equations. Thus, for conservative systems, Newtonian mechanics is a particular case of Lagrangian mechanics.

By means of a Legendre transformation which transforms functions on the vector space to functions on the dual space, al Lagrangian system of n second order differential equations is converted into a remarkably symmetrical system of $2n$ first differential equations called Hamilton's equations or canonical equations.

The fundamental structure of a Hamiltonian mechanical system has proved to be a symplectic structure. The principal cause of introduction of symplectic geometry in mechanics is to be sought in Hamilton's equations in $R^n \times R^n$.[1)]

Although the problems of the proper quadratic first integrals(QFI) have been treated by various authors [1],[2],[3], etc., the geometric and physical meaning of

[1)] R. Abraham and J. E. Marsden, Foundations of Mechanics.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX the proper QFI have remained open for a long time. We have focused our attention on this problem for symplectic manifolds, mainly on the relations between the QFI and the trajectories of a particle motion.

M. Ikeda and Y. Nishino have classified the dynamical system treated here into the three type I, II and III.

Among the proper QFI, which contain 5 independent factors, but in type I, there no proper QFI. Therefore we are led to consider the types II and III. In the present paper we have further classified the proper QFI with respect to the trajectories. For the type III, the proper QFI exhibits a simple dependence on the initial position, but not on the initial velocity. Important is the type II since we must consider the energy dependence when we treat the velocity dependence.

We may say that among the proper QFI the total energy is the only one that does not depend on any initial condition. Analysis treated here is closely related with the study of the general Killing tensors which remains open in the meaning.

2. Preliminaries

In a natural dynamical system with n-degrees of freedom, the kinetic and potential energies take the forms $T = \frac{1}{2}\eta_{ji}(x)\dot{x}^j\dot{x}^i$, $U = U(x)$, where x^i and $\dot{x}^i$ are the generalized coordinates and velocities respectively, $\eta_{ji}(x)$ and $U(x)$ functions of x^i alone. As was stated in introduction, the configuration space may be regarded as a symplectic manifold having the fundamental tensor η_{ji} .

We shall have a brief review of the results of Ref. 1, which are needed for the present arguments. The configuration space is assumed to be a symplectic manifold referred to Cartesian $x^i (i = 1, \cdots, n)$, and the potential function U is restricted to be a function of the final coordinate x^n only.[2)]

The QFI under consideration are written as the form

$$Q = \frac{1}{2}\eta_{ji}(x)\dot{x}^j\dot{x}^i + \zeta(x),\ \eta_{ji} = \eta_{ij}^{3)} \tag{2.1}$$

[3)] with the constraints

$$\nabla_k\eta_{ji} + \nabla_j\eta_{ik} + \nabla_i\eta_{kj} = 0, \tag{2.2}$$

$$\nabla_i\zeta = \eta_{in}U'(U' = \frac{dU}{dx^n}), \tag{2.3}$$

where ∇_i denotes the covariant differentiation with respect to x^i, η_{ji} and ζ, functions of x^i, are symmetries covariant tensor and scalar, respectively.

The general solution of (2.2) is given by

$$\eta_{ji} = a_{lkji}x^l x^k + a_{kji}x^k + a_{ji}, \tag{2.4}$$

where a_{lkji}, a_{kji} and a_{ji} are constants satisfying the following relations Ref. 1, Propositions 2.1.3). For a_{lkji}, we have

$$\begin{aligned}
&a_{nnnn} = a_{xnnn} = a_{nxnn} = a_{nnxn} = a_{nnnx} = 0,\ a_{xn\lambda n} = a_{\lambda nxn},\\
&a_{n\lambda nx} = a_{n\lambda xn} = a_{\lambda nnx} = a_{\lambda nxn} = -\frac{1}{2}a_{\lambda xnn} = -\frac{1}{2}a_{nn\lambda x},\\
&a_{\mu\lambda nx} = a_{\mu\lambda xn} = a_{\lambda x\mu n} = -\frac{1}{2}a_{\mu n\lambda x} = -\frac{1}{2}a_{n\mu\lambda x},\ a_{\mu\lambda xn} + a_{\lambda x\mu n} + a_{x\mu\lambda n} = 0,\\
&a_{\nu\mu\lambda x} = a_{\nu\mu x\lambda} = a_{\mu\nu\lambda x},\ a_{\nu\mu\lambda x} + a_{\nu\lambda x\mu} + a_{\nu x\mu\lambda} = 0.
\end{aligned} \tag{2.5}$$

[2)] Unless stated otherwise, small Latin indices take the values $1, \cdots, n$, Greek ones $1, 2, \cdots, n-1$ and the summation convention is adopted.

[3)] the dot denote the generalized velocities.

For a_{kji}, we have

$$a_{nnn}=0,\ a_{nnx}=a_{nxn}=-\frac{1}{2}a_{xnn},\ a_{\lambda nx}=a_{\lambda xn}=-\frac{1}{2}a_{n\lambda x},\ a_{\mu\lambda x}=a_{\mu x\lambda},$$

$$a_{\mu\lambda x}+a_{\lambda x\mu}+a_{x\mu\lambda}=0, \tag{2.6}$$

and moreover for a_{ji}, $a_{ji}=a_{ij}$. The coefficients a_{lkji}, a_{kji} and a_{ji} are classified according to the number n appearing in the indices. This number n is called the type coefficients.

From the condition (2.3) related to U, we have further restrictions on a_{lkji} such that all components a_{lkji} of type I vanish (in Ref. 1, Proposition 3.1), i.e.,

$$a_{n\mu\lambda x}=a_{\mu n\lambda x}=a_{\mu\lambda nx}=a_{\mu\lambda xn}=0 \tag{2.7}$$

and for a_{kji},

$$a_{\lambda xn}=a_{x\lambda n}. \tag{2.8}$$

Based on the elementary estimates of the coefficients, the natural dynamical system under consideration is classified into three types according to the type of U, where we assumed that $U'\neq 0$ since the case $U'=0$ is of no importance.

Type I: $U''\neq 0,\ 3U'+(x^n+k)U''\neq 0$.

Type II: $U''\neq 0,\ 3U'+(x^n+k)U''=0$.

In these types, U is given by

$$U=\frac{a}{(x^n+k)^2}+b. \tag{2.9}$$

Type III: $U''=0$.

In this case, we have

$$U=ax^n+b\quad (a\neq 0). \tag{2.10}$$

In all the above cases, a is a nonzero constant and b, k are arbitrary constants. Corresponding to the above cases, the explicit QFI are known as follows(in Ref. 1, Theorems 3.1-3).

Type I: We have

$$Q_1 = \frac{1}{2}(a_{\nu\mu\lambda x}x^\nu x^\mu + a_{\mu\lambda x} + a_{\lambda x})x^\lambda x^x + a_{nn}\{\frac{1}{2}(x^n)^2 + U(x^n)\} \tag{2.11}$$

with $\eta_{nn} = a_{nn}$, $\eta_{xn} = 0$, $\eta_{\lambda x} = a_{\nu\mu\lambda x}x^\nu x^\mu + a_{\mu x}x^\mu + a_{\lambda\mu}$. $\zeta = a_{nn}U$.

In this case, particular constraints for a's occur such that all the a's of type nonzero vanish except a_{nn}, i.e., $a_{\lambda nxn} = a_{nxn} = a_{\lambda xn} = a_{xn} = 0$. Moreover, the number of linearly independent QFI (with constant coefficients) is given by $(n^4 - n^2 + 12)/12$.

Type II: We have

$$\begin{aligned} Q_2 =& Q_1 - a_{\lambda nxn}(x^n + 2k)\dot{x}^\lambda \dot{x}^x + (2a_{\lambda nxn} + a_{nxn})(x^n + k)\dot{x}^x \dot{x}^n \\ &- (a_{\lambda nxn}x^\lambda x^x + a_{nxn}x^x)(\dot{x}^n)^2 - 2a(a_{\lambda nxn}x^\lambda x^x + a_{nxn}x^x)(x^n + k)^{-2} \end{aligned} \tag{2.12}$$

with $\eta_{nn} = -2a_{\lambda nxn}x^\lambda x^x - 2a_{nxn}x^x + a_{nn}$, $\eta_{xn} = (2a_{\lambda nxn} + a_{nxn})(x^n + k)$, $\eta_{\lambda x} = a_{\nu\mu\lambda x}x^\nu x^\mu + a_{\mu\lambda x}x^\mu - 2a_{\lambda nxn}x^n(x^n + k)$, $\zeta = a\{a_{nn} - 2(a_{\lambda nxn}x^\lambda x^x + a_{nxn}x^x)\}(x^n + k)^{-2} + ba_{nn}$.

In this case, the number of linearly independent QFI is given by $n(n+1)(n^2 - n + 6)/12$.

Type III: We have

$$Q_3 = Q_1 - a_{\lambda xn}x^n \dot{x}^\lambda \dot{x}^x + (a_{\lambda xn}x^\lambda + a_{xn})\dot{x}^\lambda \dot{x}^n + \frac{1}{2}\{a_{\lambda xn}x^\lambda x^x + a_{xn}x^x\} \tag{2.13}$$

with $\eta_{nn} = a_{nn}$, $\eta_{xn} = a_{\lambda xn}x^\lambda + a_{xn}$, $\eta_{\lambda x} = a_{\nu\mu\lambda x}x^\nu x^\mu + a_{\mu\lambda x}x^\mu - 2a_{\lambda xn}x^n + a_{\lambda x}$, $\zeta = a(\frac{1}{2}a_{\lambda xn}x^\lambda x^x + a_{xn}x^x) + a_{nn}(ax^n + b)$.

The particular constraints for a's occur; $a_{\lambda nxn} = a_{nxn} = 0$. The number of linearly independent QFI is equal to that in Type II, i.e., $n(n+1)(n^2-n+6)/12$.

3. The equation of motion in particular dynamical systems

In this section, we shall consider solutions of the equation of motion in particular dynamical systems. The equations of motion are written

$$\ddot{x}^\mu = 0, \quad \ddot{x}^n = -U', \tag{3.1}$$

which are obtained as the Euler equations from a Lagrangian

$$L = \frac{1}{2}\{(\dot{x}^\mu)^2 + (\dot{x}^n)^2\} - U(x^n). \tag{3.2}$$

The first equations in (3.1) have the solutions

$$x^\mu(t) = a_\mu t + b_\mu, \tag{3.3}$$

where a_μ and b_μ are constants.

Proposition 3.1. *The solutions of second equation in (3.1) are given as follows.*

(1) For a type II, i.e., $U = a\,(x^n+k)^{-2} + b\ (a \neq 0)$, *put* $(x^n)^2/2 + U = E^n (E^n$ *: constant). Then we have two cases according to constant* E^n*:*

(II-i) $E^n \neq b$:

$$x^n(t) = \pm\sqrt{2(E^n-b)t^2 + c_1 t + c_2} + c_3, \tag{3.4}$$

(II-ii) $E^n = b$:

$$x^n(t) = \pm\sqrt{2\sqrt{-2at} + c_4} + c_5, \tag{3.5}$$

where c_i are constants.

(2) For a type III, i.e., $U = ax^n + b$ ($a \neq 0$), we have

$$x^n(t) = -\frac{1}{2}at^2 + c_6 t + c_7, \; c_6, \; c_7 : \text{ constants.} \tag{3.6}$$

Proof. First observe that the second equation in (3.1) can be integrated as $(\dot{x})^2/2 + U = E^n$. Thus we have

$$\int \frac{dx^n}{\sqrt{-2U + 2E^n}} = dt.$$

In type II, the left-hand side becomes

$$\int \frac{(x^n + k)dx^n}{\sqrt{-2a + 2(E^n - b)(x^n + k)^2}}.$$

Thus we must consider two cases (II-i) and (II-ii). It is easy to see that the solutions are given by (3.4) and (3.5). For a type III, a little calculation immediately give the solution (3.6). Q.E.D.

Remark on type II: The constant E^n is total energy of potential function. The situations are different between type (II-i) and (II-ii). This causes further classifications of the quadratic integrals, which are not treated in type I.

We now consider the proper QFI $Q_2 - Q_1$ of type II and $Q_3 - Q_1$ of type III as follows:

$$\begin{aligned} Q_2 - Q_1 = & a_{\lambda n x n}[-x^n(x^n + 2k)\dot{x}^\lambda \dot{x}^x + 2(x^n + k)x^\lambda \dot{x}^x \dot{x}^n \\ & - 2x^\lambda x^x \{\frac{1}{2}(\dot{x}^n)^2 + a(x^n + k)^- 2\}] \\ & + a_{nxn}[(x^n + k)\dot{x}^x \dot{x}^n - 2\{\frac{1}{2}(\dot{x}^n)^2 + a(x^n + k)^{-2}\}], \end{aligned} \tag{3.7}$$

$$Q_3 - Q_1 = a_{\lambda x n}(x^\lambda \dot{x}^\lambda \dot{x}^x - x^n \dot{x}^x \dot{x}^n + \frac{a}{2}x^\lambda x^x) + a_{xn}(\dot{x}^x \dot{x}^n + ax^x), \tag{3.8}$$

where the expressions are slightly changed from (2.12) and (2.13).

From the above two expressions, we immediately know that each component of QFI is independent of time t, and is constant. Finally, we treat the type I with terms of linear first integrals and energy. From the above statements, all the potentials other than those in type II and III belong to the type I. Unlike the above two types, the potential function U can not be determined explicitly. Therefore we cannot solve the equations (3.1) and (3.2). Even if the solution is not known, it is already known that for each component of QFI, Q_1 is a constant. To illustrate the situation, we give an example.

Example. *$U(z) = \alpha z^2$, where α is a positive constant. This potential function is clearly of type I. The equations (3.1) and (3.2) with the above potential function have the following solution:*

$$x(t) = \left(m_1 t + n_1,\ m_2 t + n_2,\ \frac{\sin(\pm\sqrt{2\alpha}t) + \beta}{\sqrt{2\alpha}} \right),$$
$$v(t) = \left(m_1,\ m_2,\ \pm\cos(\pm\sqrt{2\alpha t} + \beta \right), \tag{3.9}$$

where $v(t) = \dfrac{dx}{dt}$ and m_i, n_i; β are integration constants. If we insert (3.8) and (3.9) into Q_2 or Q_3, then we know immediately that each component is not constant and depends on time t.

From this example, we know that the QFI of a certain type can never be a constant along any trajectory of the other type.

4. The meaning of proper QFI

The mathematical and physical meaning of the proper QFI remain to be an open problem. Only known quadratic first integral is total energy. This problem is closely related to the mathematical interpretations of the killing tensors η_{ij}. In order to give some interpretations, we shall study the intimate relations between these proper QFI and trajectories. The initial values of the trajectories are classified as follows. For the initial position $x_0 = x(0)$, we have two cases: (a) $x_0 = (0,0,0)$, origin and (b) $x_0 \neq (0,0,0)$, other general position.

For the initial velocity $v_0 = v(0)$, we have three cases: (c) $v_0 = (0,0,k)$, $k \neq 0$, i.e., z-direction, (d) $v_0 = (m,n,0)$, at least one of m,n is not zero, i.e., v_0 is tangent to the xy-plane and (e) $v_0 = (m,n,k)$, other than (c) or (d).

First, we shall treat type III.

Type III. From (3.8), there are components of QFI as follows:

$$F_1 = c_6 m_2 n_2 - c_7 m_2^2 + \frac{1}{2} a n_2^2,\ F_2 = -c_7 m_1^2 + c_6 m_1 n_1 + \frac{1}{2} a n_1^2,\ F_3 = c_6 m_2 + a n_2,$$
$$F_4 = (c_7 - c_6) m_1 n_1 - a n_1 n_2 + (c_7 m_1 c_6 n_1) m_2,\ F_5 = c_6 m_1 + a n_1. \tag{4.1}$$

The initial value of the trajectory from (3.6) is given by

$$x_0 = (n_1,\ n_2,\ c_7),\ v_0 = (m_1,\ m_2,\ c_6). \tag{4.2}$$

Based upon these explicit forms, we have the following proposition 4.1.

Proposition 4.1. *Three components $F_1,\ F_2,\ F_3$ of the proper QFI can be trivialized by a proper choice of the initial position x_0.*

Proof. Set x_0 at the origin, i.e., $n_1 = n_2 = c_7 = 0$. Then $F_1,\ F_2,\ F_4$ vanish, but the other F_3 and F_5 do not. Q.E.D.

Therefore we shall call $F_1,\ F_2,\ F_4$ the position-dependent QFI. Concerning the initial velocity dependence, every component of (4.1) can not be trivialized by any choice of v_0.

This proposition 4.1 asserts that QFI has less dependence on the initial velocity than the initial position Therefore we can classify the QFI in type III as follows;

$$Q = Q_3 + Q_1, \tag{4.3}$$

where Q_1 is expressed by linear first integral, and Q_3 is composed of the total energy in the z-direction.

Finally we consider the type II.

Type II. From (3.7), the components of QFI are obtained as follows:

$$\begin{aligned} H_1 &= \frac{1}{2}c_1(m_1n_2 + m_2n_2) - c_1m_1m_2 - 2(E_z - b)n_1n_2, \\ H_2 &= \frac{1}{2}(c_3^2 + c_2)m_2^2 + \frac{1}{2}c_1m_2n_2 - (E_z - b)n_2^2, \\ H_3 &= \frac{1}{2}(c_3^2 + c_2)m_1^2 + \frac{1}{2}c_1m_1n_1 - (E_z - b)n_1^2, \\ H_4 &= \frac{1}{2}c_1m_1 - 2(E_z - b), \\ H_5 &= \frac{1}{2}c_1m_2 - 2(E - z - b) \end{aligned} \tag{4.4}$$

We know the initial value of trajectory by (3.4) and (3.5) as follows:

$$x_0 = (n_1, n_2, \pm\sqrt{c_2} + c_3), \ v_0 = (m_1, m_2, c_1/\sqrt{c_2}). \tag{4.5}$$

On the basis of these explicit forms, we have the following proposition.

Proposition 4.2. *Two components H_2, H_3 of the proper QFI can be trivialized by the choi^e of the initial position x_0.*

Proof. Put x_0 at the origin, i.e., $n_1 = n_2 = \pm\sqrt{c_2} + c_3 = 0$. Then we have $H_2 = H_3 = 0$, but the other H_1, H_4 and H_5 do not vanish. Q.E.D.

Therefore we shall call H_2, H_3 the position-dependent QFI.

Next propositions are concerned with the initial velocity dependence. In this case, we must consider two cases: (i) $E_z \neq b$ and (ii) $E_z = b$.

For the first case, we have the following proposition.

Proposition 4.3. *If $E_z \neq b$, each component of (4.4) can not be trivialized by the choice of the initial position x_0.*

Similarly for the above case (ii), we have the following proposition.

Proposition 4.4. *For the case (ii) $E_z = b$, the components (4.4) of the QFI are given by*

$$H_1 = \frac{1}{2}c_1(m_1n_2 + m_2n_2) - c_1m_1m_2, \quad H_2 = \frac{1}{2}(c_3^2 + c_2)m_2^2 + \frac{1}{2}c_1m_2n_2,$$
$$H_3 = \frac{1}{2}(c_3^2 + c_2)m_1^2 + \frac{1}{2}c_1m_1n_1, \quad H_4 = \frac{1}{2}c_1m_1, \quad H_5 = \frac{1}{2}c_1m_2. \tag{4.6}$$

Proof. The above results can immediately be obtained from (4.3) with $E_z = b$. Q.E.D.

Similarly we can obtain a result concerning the initial value dependence. We have the following proposition.

Proposition 4.5.

(a) *If $v_0 = (0, 0, c_1/\sqrt{c_2})$, $c_2 \neq 0$, then all components of (4.6) of QFI vanish.*

(b) *If $v_0 = (m_1,\ m_2,\ 0)$ (at least one of m_1, m_2 is not zero), then the nontrivial components are the following two:*

$$H_2 = \frac{1}{2}(c_3^2 + c_2)m_2^2 \text{ and } H_3 = \frac{1}{2}(c_3^2 + c_2)m_1^2 \tag{4.7}$$

Thus we have shown the initial value dependence of the QFI explicitly.

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