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ON UNIVALENT SUBORDINATE FUNCTIONS

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ABSTRACT. Let $f(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots$ be regular and univalent in $\triangle = \{z : |z| < 1\}$. In this paper, using the proper subordinate functions, we investigate the some relations between subordinations and conditions of functions belonging to subclasses of univalent functions.

1. Introduction

Let

$$f(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots \tag{1.1}$$

be regular and univalent in $\triangle = \{z : |z| < 1\}$,and suppose that $f(\triangle) = D$. If $g(z)$ is regular in $\triangle$, $f(0) = g(0)$, and $g(z) \in D$, then $g(z)$ is said to be subordinate to $f(z)$ in $\triangle$, and we write [7] [8]

$$g(z) \prec f(z) \tag{1.2}$$

and also $f(z)$ is said to be superordinate to $g(z)$ in $\triangle$, (or $f(z)$ is said to be majorant to $g(z)$ in $\triangle$).

The subordinate function $g(z) \prec f(z)$ in $\triangle$ if and only if there is a function $b(z)$ that satisfies the conditions of Schwarz's Lemma $|b(z)| < |z|$ for $0 < |z| < 1$ and [8]

$$g(z) = f(b(z)) \tag{1.3}$$

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$$g(|z| < r) \in f(|z| < r) \quad |z| \leq r < 1. \tag{1.4}$$

Let $\mathcal{A}$ be the class of univalent functions $f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$ which are regular in $\triangle$.

Let ST be the subclass of $\mathcal{A}$ composing of functions which are starlike. A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be starlike if and only if $z^{-1} f(z) \neq 0$ in $\triangle$ and [4] [8], Re denotes real part,

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \text{ in } \triangle. \tag{1.5}$$

Let CV be the subclass of $\mathcal{A}$ composing of functions which are convex. A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be convex if and only if $f'(z) \neq 0$ in $\triangle$ and [4] [8]

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad \text{in } \triangle. \tag{1.6}$$

Let CC be the subclass of $\mathcal{A}$ composing of functions which are close-to-convex. A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be close-to-convex if and only if $f(z) - f(-z) \neq 0$ in $\triangle$ and [5] [8]

$$Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0 \text{ in } \triangle. \tag{1.7}$$

Let αS be the subclass of $\mathcal{A}$ composing of functions which are α-spirallike. A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be α-spirallike if and only if [6] [8]

$$Re\left\{e^{-i\alpha}\frac{zf'(z)}{f(z)}\right\} > 0 \quad \text{in } \triangle. \tag{1.8}$$

In this paper, using the proper subordinate functions, we investigate the some relations between subordinations [(3.1),(3.2),(3.3) and (3.4)] and conditions [(1.5),(1.6), (1.7) and (1.8)] of functions belonging to these subclasses (ST, CV, CC and αS) of the class of univalent functions.

2. Preliminaries

In this chapter, we investigate the relations between subordinate function and regular functions $g(z,t)$ and $b(z,t)$, depending upon a real parameter t, and then using them to obtain some results on univalent subordinate functions.

Definition 2.1. Let $b(z,t)$ be regular in $\triangle$ for $0 \le t \le 1$. Let

$$|b(z,t)| < 1 \quad \text{for } |z| < 1,\ 0 \le t \le 1,\ \text{b(z,0)=z} \tag{2.1}$$

we define

$$b(z) = \lim_{t \to 0^+} \left\{ \frac{b(z,t) - z}{zt^p} \right\} \tag{2.2}$$

where p is a positive real number. We obtain the following relation.

Theorem 2.2. *If $b(z)$ is regular in $\triangle$, Re $b(0) \ne 0$,$b(0,t) = 0$,and $0 \le t \le 1$, then $Reb(z) < 0$ in $\triangle$.*

Proof. i, Since $|b(z,t)| \le |z|$ in $\triangle$ with equality only if $b(z,t) = zexpi\theta(t)$ by Schwarz's Lemma, the function

$$m(z,t) = \frac{b(z,t) - z}{b(z,t) + z} \tag{2.3}$$

is regular and $Rem(z,t) < 0$ in $\triangle$.

ii, When $b(z,t) = zexpi\theta(t)$, Since $m(z,t) = i\tan(1/2\theta(t))$ is purely imaginary, $Re(m,t) = 0$. Thus $m(z,t)$ is regular and $Rem(z,t) \le 0$ in $\triangle$ with equality occurring only if $b(z,t) = zexpi\theta(t)$. Hence

$$\begin{aligned} Reb(z) &= Re \lim_{t \to 0^+} \left\{ \frac{b(z,t) - z}{zt^p} \right\} \\ &= Re \lim_{t \to 0^+} \left\{ \frac{b(z,t) - z}{zt^p} \cdot \frac{b(z,t) + z}{b(z,t) + z} \right\} \\ &= Re \lim_{t \to 0^+} \left\{ \frac{b(z,t) - z}{zt^p} \cdot \frac{2z}{b(z,t) + z} \right\} \\ &= Re \lim_{t \to 0^+} \left\{ \frac{2m(z,t)}{t^p} \right\} < 0 \quad \text{in } \triangle \end{aligned} \tag{2.4}$$

This completes the proof of theorem 2.2.

Definition 2.3. Let $g(z,t)$ be regular in Δ, $0 \le t \le 1$. Let $g(z,0) = f(z)$ and $g(0,t) = 0$. We define

$$g(z) = \lim_{t \to 0^+} \left\{ \frac{g(z,t) - g(z,0)}{zt^p} \right\} \tag{2.5}$$

where p is a positive real number and $g(z,t)$ is continuous from $t \to 0^+$.

We obtain the following relation.

Theorem 2.4. *If $g(z,t) \prec f(z)$ in Δ ,$0 \le t \le 1$,and $Re g(0) \neq 0$, then*

$$Re \left\{ \frac{f'(z)}{g(z)} \right\} < 0 \quad in\ \Delta. \tag{2.6}$$

Proof. Since $g(z,t) \prec f(z)$ in Δ we have $g(z,t) = f\{b(z,t)\}$ in Δ, $0 \le t \le 1$, where $b(z,t)$ is regular, continuous and $|b(z,t)| \le 1$ in Δ, $0 \le t \le 1$. Since $g(z,0) = f(z)$ and $f(z)$ is univalent in Δ we have $b(z,0) = z$. Since $f(0) = 0, g(0,t) = 0$ and $f(z)$ is univalent in Δ we have $b(0,t) = 0$. Now we write

$$\begin{aligned} \frac{g(z,t) - g(z,0)}{zt^p} &= \frac{f\{b(z,t)\} - f\{b(z,0)\}}{zt^p} \cdot \frac{b(z,t) - b(z,0)}{b(z,t) - b(z,0)} \\ &= \frac{f\{b(z,t)\} - f\{b(z,0)\}}{b(z,t) - b(z,0)} \cdot \frac{b(z,t) - b(z,0)}{zt^p} \end{aligned} \tag{2.7}$$

$$\lim_{t \to 0^+} \left\{ \frac{g(z,t) - g(z,0)}{zt^p} \right\} = \lim_{t \to 0^+} \frac{f\{b(z,t)\} - f\{b(z,0)\}}{b(z,t) - b(z,0)} \cdot \lim_{t \to 0^+} \frac{b(z,t) - b(z,0)}{zt^p}$$

By (2.2), (2.5) and $b(z,0) = z$

$$g(z) = f'(z) \cdot b(z), \quad f'(z) \neq 0. \tag{2.8}$$

Therefore $b(z) = g(z)/f'(z)$, $Re b(0) = Re g(0)$. By $Re b(0) = Re g(0)$ and theorem 2.2 we have

$$Re \frac{g(z)}{f'(z)} = Re b(z) \le 0 \ \text{ in } \Delta. \tag{2.9}$$

When $g(z)$ is regular in $\triangle$ and $g(0) \neq 0$. we have

$$Re\left\{\frac{f'(z)}{g(z)}\right\} < 0 \text{ in } \triangle.$$

This completes the proof of theorem 2.4.

3. Some Result

In this chapter, using the preliminaries, we investigate some results which there exists each condition of the subclass($ST, CV, CC$and αS) of the class of univalent functions for each given proper relation subordinated to univalent functions that $f(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots$ are regular and univalent in $\triangle$.

Theorem 3.1. *Let*

$$(1-t)f(z) \prec f(z),\ 0 \leq t \leq 1, \quad in\ \triangle \tag{3.1}$$

then

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \ \ and\ f(z) \in ST\ in\ \triangle.$$

Proof. From (3.1) since $(1-t)f(z) \prec f(z), 0 \leq t \leq 1$, letting

$$g(z) = (1-t)f(z) = f\{b(z,t)\}$$

and in (2.5) taking $p = 1$ we have

$$g(z) = \lim_{t\to 0^+} \frac{g(z,t) - g(z,0)}{zt} = \frac{1}{z}\frac{\partial g(z,t)}{\partial t} = \frac{1}{z}(-f(z)) = -\frac{f(z)}{z} \neq 0$$

From (2.6), since

$$Re\left\{\frac{f'(z)}{g(z)}\right\} < 0 \text{ in } \triangle$$

we have

$$Re\left\{\frac{f'(z)}{g(z)}\right\} = Re\left\{\frac{f'(z)}{-f(z)/z}\right\} = Re\left\{-\frac{zf'(z)}{f(z)}\right\} < 0 \text{ in } \triangle.$$

Hence

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0$$

and from (1.5) $f(z) \in ST$ in $\triangle$.

This completes the proof of theorem 3.1.

Theorem 3.2. *Let*

$$\frac{1}{2}\left\{f(e^{it}z) + f(e^{-it}z)\right\} \prec f(z), \quad 0 \le t \le 1, in\ \triangle, \tag{3.2}$$

then

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \ \ and\ f(z) \in CV \ in\ \triangle.$$

Proof. From (3.2), since $\frac{1}{2}\left\{f(e^{it}z) + f(e^{-it}z)\right\} \prec f(z), 0 \le t \le 1, in\triangle$, letting

$$g(z,t) = f(e^{it}z) + f(e^{-it}z)$$

and in (2.5) taking $p = 2$ and using L'Hospital rule we have

$$\begin{aligned} g(z) &= \lim_{t\to 0+} \frac{g(z,t) - g(z,0)}{zt^2} = \lim_{t\to 0+} \frac{\frac{\partial g(z,t)}{\partial t}}{2zt} \\ &= \frac{1}{2z}\frac{\partial^2 g(z,0)}{\partial t^2} = -\left\{zf''(z) + f'(z)\right\} \end{aligned}$$

since, f' and f'' denote $\frac{\partial f}{\partial t}$ and $\frac{\partial^2 f}{\partial t^2}$ respectively,

$$\begin{aligned} \frac{\partial g(z,t)}{\partial t} &= f'(e^{it}z)ize^{it} + f'(e^{-it}z)ze^{-it}(-i) \\ &= iz\{f'(e^{it}z)e^{it} - f'(e^{-it}z)e^{-it}\} \end{aligned}$$

$$\frac{\partial^2 g(z,t)}{\partial t^2} = iz[\{f''(e^{it}z)ze^{it}ie^{it} + f'(e^{it}z)e^{it}i\} - \{f''(e^{-it}z)e^{-it}z(-i)e^{-it} + f'(e^{-it}z)e^{-it}(-i)\}]$$

$$\begin{aligned}\frac{\partial^2 g(z,0)}{\partial t^2} &= iz[\{f''(z)zi + f'(z)i\} - \{f''(z)z(-i) + f'(z)(-i)\}] \\ &= -z\{f''(z)z + f'(z)\} - z\{f''(z)z + f'(z)\} \\ &= -2z\{f''(z)z + f'(z)\}\end{aligned}$$

From (2.6), since

$$Re\left\{\frac{f'(z)}{g(z)}\right\} < 0 \text{ in } \triangle$$

we have

$$Re\left\{\frac{f'(z)}{g(z)}\right\} = Re\left\{\frac{f'(z)}{-\{zf''(z) + f'(z)\}}\right\} < 0 \text{ in } \triangle.$$

Since $f'(0) = 1, g(0) = -1$ and $Re g(0) \neq 0$

$$Re\left\{\frac{1}{1 + z\frac{f''(z)}{f'(z)}}\right\} > 0 \text{ in } \triangle.$$

Hence

$$Re\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > 0 \text{ and from (1.6) } f(z) \in CV \text{ in } \triangle.$$

This completes the proof of theorem 3.2.

Theorem 3.3. *Let*

$$(1-t)f(z) + tf(-z) \prec f(z), \; 0 \leq t \leq 1, \quad in \; \triangle \tag{3.3}$$

then

$$Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0 \; and \; f(z) \in CC \; in \; \triangle$$

Proof. From (3.3) since $(1-t)f(z)+tf(-z) \prec f(z), 0 \le t \le 1$, in $\triangle$, letting

$$g(z,t) = (1-t)f(z) + tf(-z)$$

and in (2.5) taking $p = 1$. We have

$$g(z) = \lim_{t \to 0^+} \frac{g(z,t) - g(z,0)}{zt} = \frac{1}{z}\frac{\partial g(z,t)}{\partial t} = \frac{1}{z}\{-f(z) + f(-z)\}$$

and $g(0) \neq 0$. From (2.6),since

$$Re\left\{\frac{f'(z)}{g(z)}\right\} < 0 \text{ in } \triangle$$

we have

$$Re\left\{\frac{f'(z)}{g(z)}\right\} = Re\left\{\frac{zf'(z)}{-f(z)+f(-z)}\right\} < 0 \text{ in } \triangle.$$

Hence $Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\} > 0$ and from (1.7) $f(z) \in CC$ in $\triangle$.

This completes the proof of theorem 3.3.

Theorem 3.4. *Let*

$$(1-te^{i\alpha})f(z) \prec f(z), \quad in \ \triangle \tag{3.4}$$

where $0 \le t \le 1$, *α is a real constant and* $|\alpha| < \frac{\pi}{2}$, *then*

$$Re\left\{e^{-i\alpha}\frac{zf'(z)}{f(z)}\right\} > 0 \quad and\ f(z) \in \alpha S\ in\ \triangle.$$

Proof. From (3.4) since $(1-te^{i\alpha})f(z) \prec f(z), in \triangle$, letting

$$g(z,t) = (1-te^{i\alpha})f(z)$$

and in (2.5) taking $p = 1$ we have

$$g(z) = \lim_{t \to 0^+} \frac{g(z,t) - g(z,0)}{zt} = \frac{1}{z}\frac{\partial g(z,t)}{\partial t} = \frac{1}{z}\{-e^{i\alpha}f(z)\}$$

and from (2.6), since

$$Re\left\{\frac{f'(z)}{g(z)}\right\} < 0 \text{ in } \triangle$$

we have

$$Re\left\{\frac{f'(z)}{g(z)}\right\} = Re\left\{\frac{f'(z)}{-e^{i\alpha}f(z)/z}\right\} = Re\left\{\frac{-zf'(z)}{e^{i\alpha}f(z)}\right\} < 0 \text{ in } \triangle.$$

Hence

$$Re\left\{e^{-i\alpha}\frac{zf'(z)}{f(z)}\right\} > 0$$

and from (1.8) $f(z) \in \alpha S$ in $\triangle$.

This completes the proof of theorem 3.4.

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