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ATTRACTIVITY AND ASYMPTOTIC STABILITY IN DYNAMICAL POLYSYSTEMS

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ABSTRACT. We investigate that if M is a compact subset of c-first countable space X, then $x \in A_u(M)$ if and only if $J(x) \neq \phi \subset M$.

1. Introduction

N. Kalouptsidis, A. Bacciotti and J. Tsinias have extend the properties of stability, limit set, prolongational limit, attractivity and asymptotic stability reffered to [1] for dynamical polysystems.

In this section the previous results studied for the multivalued map Γ in [2] and [5] are examined for the reachable map R.

Throughout this paper the space X will be locally compact metric space, unless otherwise restricted.

A dynamical system on X is a continuous map $\pi : X \times \mathbb{R} \to X$ with the following properties:

(i) $\pi(x, 0) = x$ for all $x \in X$,

(ii) $\pi(\pi(x, s), t) = \pi(x, s + t)$ for all $x \in X$ and $s, t \in \mathbb{R}$.

We call a family of dynamical systems $\{\pi_i | i \in I\}$ a dynamical polysystem on X. Dynamical polysystems arise in control theory.

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If 2^X denotes the set of all subsets of X and $\mathbb{R}^+$ the set of nonnegative real numbers, the reachable map of the polysystem $\{\pi_i | i \in I\}$ is the multivalued map $R : X \times \mathbb{R}^+ \to 2^X$ defined by

$$R(x,t) = \{y \in X |\text{ there exist an integer } n, t_1, \ldots, t_n \in \mathbb{R}^+ \text{ and } i_1, \ldots, i_n \in I \text{ such that} \sum_{i=1}^{n} t_i = t \text{ and } y = \pi_{i_n}(\pi_{i_{n-1}}(\ldots \pi_{i_2}(\pi_{i_1}(x,t_1),t_2),\ldots,t_{n-1}),t_n)\}.$$

For $A \subset X$ and $t \in \mathbb{R}^+$, we let $R(A,t) = \bigcup_{x \in A} R(x,t)$. Also, we define $R(x, \mathbb{R}^+)$ by $R(x)$. For $A \subset X$, we let $R(A) = \bigcup_{x \in A} R(x)$. We recall some definitions from [5].

Definition 1.1. A map R is called a *cluster map* if $DR = R$.

Definition 1.2. Let $\phi \neq A \subset X$. A map R is *uniformly bounded* on A if, for each $x \in A$, there is a neighborhood U of x such that $\overline{R(U, \mathbb{R}^+)}$ is compact.

Definition 1.3. A map R is *upper semicontinuous* at $(x,t) \in X \times \mathbb{R}^+$ if for any sequences $(x_n, t_n) \to (x,t)$ and $y_n \in R(x_n, t_n)$, there is a sequence $z_n \in R(x,t)$ such that $d(y_n, z_n) \to 0$. A map R is upper semicontinuous on $A \subset X$ if for all $(x,t) \in A \times \mathbb{R}^+$, it is upper semicontinuous at (x,t).

Proposition 1.4. *A map R is upper semicontinuous at $(x,t) \in X \times \mathbb{R}^+$ if and only if for each $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$R(B(x,\delta),(t-\delta,t+\delta)) \subset B(R(x,t),\varepsilon).$$

Proof. To prove the necessary condition, assume that there is a $\varepsilon > 0$ such that for each $\delta > 0$,

$$R(B(x,\delta),(t-\delta,t+\delta)) \not\subset B(R(x,t),\varepsilon).$$

Then, for each integer n, $R(B(x,\frac{1}{n}),(t-\frac{1}{n},t+\frac{1}{n})) \not\subset B(R(x,t),\varepsilon)$. Thus there is a sequence $y_n \in R(B(x,\frac{1}{n}),(t-\frac{1}{n},t+\frac{1}{n}))$ such that $y_n \notin B(R(x,t),\varepsilon)$. We can choose sequences $x_n \in B(x,\frac{1}{n}), t_n \in (t-\frac{1}{n},t+\frac{1}{n})$ so that $y_n \in R(x_n,t_n)$, $x_n \to x$ and

$t_n \to t$. Since R is upper semicontinuous at (x,t), there is a sequence $z_n \in R(x,t)$ such that $d(y_n, z_n) \to 0$. We can choose an integer m so that $d(y_m, z_m) < \varepsilon$. It follows that $d(y_m, R(x,t)) \leq d(y_m, z_m) < \varepsilon$. Clearly, $y_m \in B(R(x,t), \varepsilon)$. This is a contradiction. Hence for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$R(B(x,\delta), (t-\delta, t+\delta)) \subset B(R(x,t), \varepsilon).$$

Next, we shall show the sufficient condition. Suppose that R is not upper semicontinuous at (x,t). Then there are sequences $(x_n, t_n) \to (x,t), y_n \in R(x_n, t_n)$ and $\varepsilon > 0$ such that for each $n, d(y_n, R(x,t)) \geq \varepsilon$. By assumption, there is a $\delta > 0$ such that

$$R(B(x,\delta), (t-\delta, t+\delta)) \subset B(R(x,t), \varepsilon).$$

Also, there is an integer m such that $x_m \in B(x,\delta), t_m \in (t-\delta, t+\delta)$. Thus we have $y_m \in R(x_m, t_m) \subset R(B(x,\delta), (t-\delta, t+\delta)) \subset B(R(x,t), \varepsilon)$. Clearly, $d(y_m, R(x,t)) < \varepsilon$.This contradiction shows that R is upper semicontinuous at (x,t). The Proposition is proved.

Proposition 1.5. *If R is uniformly bounded on $A \subset X$ and cluster, then R is upper semicontinuous on A.*

Proof. Assume that R is not upper semicontinuous at (x,t). Then there are sequences $(x_n . t_n) \to (x,t), y_n \in R(x_n, t_n)$ and $\varepsilon > 0$ such that for each $n, d(y_n, R(x,t))$ $\geq \varepsilon$. By hypothesis, there is a neighborhood U of x such that $\overline{R(U, \mathbb{R}^+)}$ is compact. We may assume that $x_n \in U$. Thus we have $y_n \in R(x_n, t_n) \subset R(U, \mathbb{R}^+)$. Since $\overline{R(U, \mathbb{R}^+)}$ is compact, there is a sequence $y_n \to y$. We can find an integer m with $d(y_m, y) < \varepsilon$. It follows that $y \in DR(x,t)$. Since R is a cluster map, we have $d(y_m, R(x,t)) \leq d(y_m, y) < \varepsilon$. This contradicts the fact that $d(y_m, R(x,t)) \geq \varepsilon$. The proposition is completed.

2. Attractivity and Asymptotic Stability

In this section the concept of attractivity and asymptotic stability are described and are characterized in terms of the basic multivalued map D, J and Λ.

Let M be a compact subset of X.

Definition 2.1. A set M is stable if for any neighborhood U of M, there exists a neighborhood V of M such that $R(V) \subset U$.

Definition 2.2. *The region of attraction* of M is denoted by $A(M)$ and is defined by

$$A(M) = \{x \in X \mid \text{there exists } t \in \mathbb{R}^+ \text{ such that } R(x, [t, \infty)) \subset U \\ \text{for any neighborhood } U \text{ of } M\}.$$

Definition 2.3. A set M is called an *attractor* if $A(M)$ contains a neighborhood of M.

Definition 2.4. *The region of uniform attraction* $A_u(M)$ is defined by

$$A_u(M) = \{x \in X \mid \text{there exists a neighborhood } V \text{ of } x \text{ and } t \in \mathbb{R}^+ \\ \text{such that } R(V, [t, \infty)) \subset U \text{ for any neighborhood } U \text{ of } M\}.$$

Definition 2.5. A set M is called *an uniform attractor* if $A_u(M)$ contains a neighborhood of M.

Definition 2.6. A set M is *asymptotically stable* if M is stable and an attractor.

Proposition 2.7. *Let M be a compact subset of X. Then M is stable if and only if for any neighborhood U of M, there is a compact positively invariant neighborhood V of M such that $V \subset U$.*

Proof. Suppose that M is stable. We can choose a comapct neighborhood W of M so that $W \subset U$. By stability of M, there is a neighborhood A of M such that $R(A, \mathbb{R}^+) \subset W$. Let $V = \overline{R(A, \mathbb{R}^+)}$. Then V is a compact positively invariant neighborhood of M. Thus the necessary condition is proved. The proof of the reverse is straightforward.

Proposition 2.8. *Let M be a compact subset of c-first countable space X. Then $x \in A(M)$ if and only if $\Lambda(x) \neq \phi \subset M$.*

Proof. Suppose that $x \in A(M)$. Take a compact neighborhood U of M. Then there is $s \in \mathbb{R}^+$ such that $R(x, [s, \infty)) \subset U$. Since $\overline{R(x, [s, \infty))}$ is compact, $\Lambda(x) \neq \phi$. We claim that $\Lambda(x) \subset M$. Suppose that $\Lambda(x) \not\subset M$. Then there is $y \in \Lambda(x)$ such that $y \notin M$. We choose a neighborhood V of M such that $y \notin \overline{V}$. Thus there is $r \in \mathbb{R}^+$ such that $R(x, [r, \infty)) \subset V$. We have $\Lambda(x) \subset \overline{R(x, [r, \infty))} \subset \overline{V}$ and $y \in \overline{V}$. This is a contradiction. Hence $\Lambda(x) \subset M$. Next, we show that the reverse holds. Let U be any neighborhood of M. Then U is any neighborhood of $\Lambda(x)$. By compactness of M, $\Lambda(x)$ is compact. Thus there exists $s \in \mathbb{R}^+$ such that $R(x, [s, \infty)) \subset U$. It follows that $x \in A(M)$. Hence the proposition is complete.

A similar description holds for the region of uniform attraction $A_u(M)$ as the next proposition indicates.

Proposition 2.9. *Suppose that M is a compact subset of c-first countable space X. Then $x \in A_u(M)$ if and only if $J(x) \neq \phi \subset M$.*

Proof. Let $J(x)$ be a nonempty subset of M. Then it is compact. Suppose U is any neighborhood of M. Since $J(x) \subset M, U$ is any neighborhood of $J(x)$. Thus there exists a neighborhood V of x and $t \in \mathbb{R}^+$ such that $R(V, [t, \infty)) \subset U$. Hence we have $x \in A_u(M)$.

Conversely, let $x \in A_u(M)$. We choose a neighborhood U of M such that $\overline{U}$ is compact. Therefore, there exists a neighborhood V of x and $t \in \mathbb{R}^+$ such that $R(V, [t, \infty)) \subset U$. Since $\overline{R(x, [t, \infty))} \subset \overline{R(V, [t, \infty))} \subset \overline{U}$, $\overline{R(x, [t, \infty))}$ is compact. Thus $\Lambda(x) \neq \phi$. Clearly, $J(x) \neq \phi$. Next we show that $J(x) \subset M$. Suppose that $J(x) \not\subset M$. We choose $y \in J(x) - M$. Then there exists a neighborhood W of M such that $y \notin \overline{W}$. Thus there exists a neighborhood V of x and $t \in \mathbb{R}^+$ such that $R(V, [t, \infty)) \subset W$. We clearly have $y \in J(x) \subset \overline{R(V, [t, \infty))} \subset \overline{W}$. This contradicts the fact that $y \in \overline{W}$. Hence $J(x) \subset M$. The proposition is proved

Proposition 2.10. *Let M be a comapct subset of X and suppose M is stable. Then there is a neighborhood W of M such that a cluster map R is uniformly bounded on*

W.

Proof. We can choose a compact neighborhood U of M. By assumption, there is a $\delta > 0$ such that $R(B(M,2\delta),\mathbb{R}^+) \subset U$. Let $W = B(M,\delta)$. For each $x \in W$, we have $R(B(x,\delta),\mathbb{R}^+) \subset R(B(W,\delta),\mathbb{R}^+) \subset R(B(M,2\delta),\mathbb{R}^+) \subset U$. Thus $\overline{R(B(x,\delta),\mathbb{R}^+)}$ is compact. Thus the proposition is proved.

Proposition 2.11. *Let a compact subset M of X be asymptotically stable. Then M is a uniform attractor.*

Proof. By proposition 1.5 and 2.10, there is a neighborhood W of M such that $W \subset A(M)$ and R is upper semicontinuous on W. Let $x \in W$. Since M is stable, for any neighborhood U of M, there is $\varepsilon > 0$ such that $R(B(M,2\varepsilon),\mathbb{R}^+) \subset U$. Clearly, $x \in A(M)$. Thus there is $s \in \mathbb{R}^+$ such that $R(x,[s,\infty)) \subset B(M,\varepsilon)$. Since R is upper semicontinuous at (x,s), there is a neighborhood V of x such that $R(V,s) \subset B(R(x,s),\varepsilon) \subset B(B(M,\varepsilon),\varepsilon) \subset B(M,2\varepsilon)$. For each $y \in R(V,[s,\infty))$, there is $z \in V$ and $t \geq s$ such that $y \in R(z,t)$. From the fact that $R(z,t) = R(R(z,s),t-s) \subset R(R(V,s),t-s) \subset R(B(M,2\varepsilon),\mathbb{R}^+) \subset U$, we have $y \in U$ and so $R(V,[s,\infty)) \subset U$. It follows that $x \in A_u(M)$. We have $W \subset A_u(M)$. Hence M is a uniform attractor.

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