ON THE AXIOM OF CHOICE IN A WELL-POINTED TOPOS

IG SUNG KIM

ABSTRACT. Topos is a set-like category. For an axiom of choice in a topos, F. W. Lawvere and A. M. Penk introduced another versions of the axiom of choice. Also it is showed that general axiom of choice and Penk's axiom of choice are weaker than Lawvere's axiom of choice. In this paper we study that weak form of axiom of choice, axiom of choice, Penk's axiom of choice and Lawvere's axiom of choice are all equivalent in a well pointed topos.

1. Introduction

The concept of a topos as a system was developed by F. W. Lawvere and M. Tierney. The most prominent example of an elementary topos is a category Set of sets and maps. In particular, Lawvere [4] introduced an axiom of choice as following form: for any noninitial object A and $f : A \to B$, there exists a morphism $g : B \to A$ such that $f \circ g \circ f = f$. And A. M. Penk [5] introduced an axiom of choice as following another form: for any noninitial object A, there exists $\sigma : \Omega^A \to A$ such that for all $f : 1 \to \Omega^A$, we have $\sigma \circ f \in f'$ where $f' : A' \to A$ is a monomorphism, provided that $ev \circ (f \times i_A)$ is not the characteristic morphism of $0 \to A$, which is weaker form than Lawvere's axiom. In a topos Set, three forms (general form and the previous two forms) of the axiom of choice are all equivalent. In this paper, we study various forms of the axiom of choice in a well-pointed topos.

Definition 1.1. An *elementary topos* is a category $\mathcal{E}$ that satisfies the following:

(T1) $\mathcal{E}$ is finitely complete,

(T2) $\mathcal{E}$ has exponentiation,

(T3) $\mathcal{E}$ has a subobject classifier.

(T2) means that for every object A in $\mathcal{E}$, endofunctor $(-) \times A$ has its right adjoint $(-)^A$. Hence for every object A in $\mathcal{E}$, there exists an object B^A, and a morphism $ev_A : B^A \times A \rightarrow B$, called the evaluation map of A, such that for any Y and $f : Y \times A \rightarrow B$ in $\mathcal{E}$, there exists a unique g such that $ev_A \circ (g \times i_A) = f$. And subobject classifier in (T3) is an $\mathcal{E}$-object Ω, together with a morphism $true : 1 \rightarrow \Omega$ (denoted by the letter "$\top$") such that for any monomorphism $h : D \rightarrow C$, there is precisely one map $C \rightarrow \Omega$, called the characteristic map of $h : D \rightarrow C$ (denoted by χ_h) that makes the following diagram a pullback:

$$
\begin{array}{ccc}
D & \longrightarrow & 1 \\
\scriptstyle h \downarrow & & \downarrow \scriptstyle \top \\
C & \underset{\chi_h}{\longrightarrow} & \Omega
\end{array}
$$

Definition 1.2. (1) A topos is called *degenerate* if there is a morphism $x : 1 \rightarrow 0$ where 0 and 1 are an initial and a terminal object respectively, that is, all objects are isomorphic.

(2) A non-degenerate topos is called *well-pointed* if it satisfies the extentionality principle for arrow, that is, if $f, g : A \rightarrow B$ are a pair of distinct parallel arrows, then there is an element $a : 1 \rightarrow A$ of A such that $f \circ a \neq g \circ a$.

(3) A non-degenerate topos is called *bivalent* if $\Omega = \{\top, \bot\}$.

Definition 1.3. A topos is called *Boolean* if for every object D,

$(\mathrm{Sub}(D), \in)$ is a Boolean algebra where $\mathrm{Sub}(D)$ is the class of monomorphisms with common codomain D, and we say $g \in f$ if there exists a morphism $h : B \rightarrow A$ such that $f \circ h = g$ where $f : A \rightarrow D$ and $g : B \rightarrow D$ are monomorphisms.

Definition 1.4. Let A be an $\mathcal{E}$-object.

(1) The *support* of an object A is defined to be the subobject m : supp $(A) \to 1$ of the terminal object 1 given by the epi-monic factorization [2] of the unique arrow ! : $A \to 1$ such that ! $= m \circ e$ where $e : A \to$ supp (A) is an epimorphism.

(2) We say that *supports split* (SS) in a topos $\mathcal{E}$ if, for every $\mathcal{E}$-object A the epic part $e : A \to$ supp (A) of the epi-monic factorization of ! : $A \to 1$ is a retraction.

Lemma 1.5. $\mathcal{E}$ *is well-pointed if and only if $\mathcal{E}$ is Boolean, bivalent, and has spliting supports.*

Proof. [1, Chapter 12].

Lemma 1.6. *In a well-pointed topos, every object is an injective object.*

Proof. For any monomorphism $m : A \to B$ and any morphism $f : A \to J$, since it is Boolean, there exists a monomorphism $-m : -A \to B$ such that

$$
\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow & & \downarrow{\scriptstyle m} \\
-A & \xrightarrow{\ -m\ } & B
\end{array}
$$

is a pushout diagram.

For $f : A \to J$ and $b \circ ! : -A \to J$ where ! : $-A \to 1$ and $b : 1 \to J$,

$$
\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow & & \downarrow{\scriptstyle f} \\
-A & \xrightarrow{\ bo!\ } & J
\end{array}
$$

commutes.

Hence there exists a morphism $t : B \to J$ such that $t \circ m = f$.

Definition 1.7.

(1) We say that an object A of a topos $\mathcal{E}$ is *internally projective* if the functor $(-)^A : \mathcal{E} \to \mathcal{E}$ preserves epimorphisms.

(2) We say that an epimorphism $u : C \to D$ is *locally split* in $\mathcal{E}$ if there exists an object V with an epimorphism $V \to 1$ such that $V^*(u)$ is a retraction in $\mathcal{E}/V$.

(3) We say that $\pi_Y(f)$ has *global support* if there exists an epimorphism $e : \pi_Y(f) \to 1$ where $f : X \to Y$ is a morphism and $\pi_Y(f)$ is a pullback of $f^Y : X^Y \to Y^Y$ and $i_Y : Y \to Y$.

2. On the Axiom of Choice in a Well-Pointed Topos

In this section we study equivalent forms of the axiom of choice in a well-pointed topos.

Proposition 2.1. *For any topos $\mathcal{E}$, the following statements are equivalent:*

(1) Every object of $\mathcal{E}$ is internally projective.

(2) Every epimorphism in $\mathcal{E}$ is locally split.

(3) If $e : A \to B$ is an epimorphism in $\mathcal{E}$, then $\Pi_B(e)$ has global support.

Proof. (1) $\Rightarrow$ (3): Let $e : A \to B$ be an epimorphism. By hypothesis, $e^B : A^B \to B^B$ is also an epimorphism. Since

$$
\begin{array}{ccc}
\pi_B(e) & \longrightarrow & A^B \\
\downarrow & & \downarrow{\scriptstyle e^B} \\
1 & \xrightarrow{\quad i_B \quad} & B^B
\end{array}
$$

is a pullback, it yields an epimorphism $\pi_B(e) \to 1$. Thus $\pi_B(e)$ has global support.

(3) $\Rightarrow$ (2): Let $e : A \to B$ be an epimorphism. By hypothesis, there exists an object $\pi_B(e)$ such that $\pi_B(e) \to 1$ is an epimorphism. We only show that $\pi_B(e)^*(e)$ is a retraction in $\mathcal{E}/\pi_B(e)$. By the definition of $\pi_B(e)^*(e)$, $p_2 \circ \pi_B(e)^*(e) = \pi_2$. By the product $A \times \pi_B(e)$, there exists a morphism $k : B \times \pi_B(e) \to A \times \pi_B(e)$ such that $\pi_2 \circ k = p_2$,

$$
\begin{array}{ccc}
\pi_B(e) & \xleftarrow{\;\;i\;\;} & \pi_B(e) \\
\pi_2 \uparrow & & \uparrow p_2 \\
A \times \pi_B(e) & \xleftarrow{\;\;k\;\;} & B \times \pi_B(e) \\
\pi_1 \downarrow & & \downarrow p_1 \\
A & \xleftarrow[\;s\;]{} & B
\end{array}
$$

where s is a right inverse of e by $hom(B^*(1), e) \cong hom(1, \pi_B(e))$. Hence $p_2 \circ (\pi_B(e)^*(e) \circ k) = p_2$. Since p_2 is monic, it yields $\pi_B(e)^*(e)$ is a retraction in $\mathcal{E}/\pi_B(e)$.

(2) $\Rightarrow$ (1): Let $u : C \to D$ be an epimorphism. By hypothesis, there exists an object V with an epimorphism $V \to 1$ such that $V^*(u)$ is a retraction in $\mathcal{E}/V$. Then $V^*(u^X)$ is isomorphic to $V^*(u)^{V^*(X)}$ for any object X [3]. Since $V^*(u^X)$ is an epimorphism and V^* reflects epimorphisms [3], it yields that u^X is an epimorphism.

Theorem 2.2. *In a well-pointed topos* $\mathcal{E}$, *the following statements are equivalent :*

(1) Every epimorphism is a retraction.

(2) For any noninitial object A and $f : A \to B$, there exists a morphism $g : B \to A$ such that $f \circ g \circ f = f$.

(3) For any noninitial object A, there exists $\sigma : \Omega^A \to A$ such that for all $f : 1 \to \Omega^A$, we have $\sigma \circ f \in f'$ where $f' : A' \to A$ is a monomorphism, provided that $ev \circ (f \times i_A)$ is not the characteristic morphism of $0 \to A$.

Proof. (1) $\Rightarrow$ (2): Let $f : A \to B$ be a morphism. Then we have $f = m \circ e$ where $e : A \to C$ is an epimorphism and $m : C \to B$ is a monomorphism. By LEMMA 1.6,

there exists a morphism $r : B \to C$ such that $r \circ m = i_C$ and by hypothesis, there exists a morphism $s : C \to A$ such that $e \circ s = i_C$. Let $g = s \circ r : B \to A$. Then it turns out that $f \circ (g \circ f) = f$,

since $(m \circ e) \circ ((s \circ r) \circ (m \circ e)) = m \circ (e \circ s) \circ (r \circ m) \circ e = m \circ e = f$.

(2) $\Rightarrow$ (1): If $f : A \to B$ is an epimorphism where $A \cong 0$, then f is monic, hence f is an isomorphism, so is split by its inverse. If $A \not\cong 0$, by hypothesis we get $g : B \to A$ such that $f \circ g \circ f = f = i_B \circ f$. Since f is an epimorphism, we get $f \circ g = i_B$, making $g : B \to A$ a right inverse of $f : A \to B$.

(1) $\Rightarrow$ (3): Consider the product object $\Omega^A \times \Omega$ together with two projections $p_2' : \Omega^A \times \Omega \to \Omega$ and $p_1' : \Omega^A \times \Omega \to \Omega^A$. Then by the definition of product, for each $ev : \Omega^A \times A \to \Omega$ and $p_1 : \Omega^A \times A \to \Omega^A$, there exists a morphism $< p_1, ev >: \Omega^A \times A \to \Omega^A \times \Omega$, where $ev \circ < c, a >= \top$ if $a \in b$ for two morphisms $c : 1 \to \Omega^A$, $a : 1 \to A$ and a monomorphism $b : B \to A$. And for each $t : \Omega^A \to \Omega$ and $i : \Omega^A \to \Omega^A$ where $t \circ d = \top$ and $i_{\Omega^A} \circ d = d$ for all $d : 1 \to \Omega^A$, there exists a morphism $< i_{\Omega^A}, t >: \Omega^A \to \Omega^A \times \Omega$.

Since $\mathcal{E}$ is bivalent, ev is an epimorphism. Hence it yields that $< p_1, ev >$ is an epimorphism. By hypothesis, there exists a morphism $h : \Omega^A \times \Omega \to \Omega^A \times A$ such that $< p_1, ev > \circ h = i_{\Omega^A \times \Omega}$.

We construct a morphism $p_2 \circ h \circ < i_{\Omega^A}, t >: \Omega^A \to A$ where $p_2 : \Omega^A \times A \to A$. We show that $p_2 \circ h \circ < i_{\Omega^A}, t > \circ c \in b$ where $c : 1 \to \Omega^A$ is any morphism and $b : B \to A$ is a monomorphism. Since

$$h \circ < i_{\Omega^A}, t > \circ c = h \circ < c, \top >=< d, a >: 1 \to \Omega^A \times A,$$

it yields $d = p_1 \circ \ < d, a >= p_1 \circ \ h \circ < c, \top >= p_1' \circ < p_1, ev > \circ h < c, \top >= p_1' \circ < c, \top >= c$.

We only show that $a \in b$, that is, $ev \circ < c, a >= \top$. Since $ev \circ < c, a >= ev \circ \ h \circ < c, \top >= ev \circ \ h \circ < p_1, ev > \circ < c, b \circ x >$ where $x : 1 \to B$ and $ev \circ \ h \circ < p_1, ev > \circ < c, b \circ x >= p_2' \circ < p_1, ev > \circ h \circ \ < p_1, ev > \circ < c, b \circ x >= ev \circ < c, b \circ x >= \top$, it implies $a \in b$. Therefore it turns out that $p_2 \circ h \circ < i_{\Omega^A}, t > \circ c = p_2 \circ < d, a >= a \ \in \ b$.

$(3) \Rightarrow (1)$: Let $e : X \to Y$ be an epimorphism and $a : 1 \to X$ be any morphism. By hypothesis, we construct following morphism
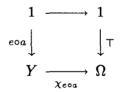
$$\sigma \circ (\Omega^e \circ \{\}) : Y \to X$$

where $\{\} : Y \to \Omega^Y$ is a singleton map.

$$
\begin{array}{ccc}
X & \xrightarrow{\ e\ } & Y \\
\sigma \uparrow & & \downarrow \{\} \\
\Omega^X & \xleftarrow[\Omega^e]{} & \Omega^Y
\end{array}
$$

We need only show that $e \circ \sigma \circ \Omega^e \circ \{\} \circ e \circ a = e \circ a$.

Since $e \circ a : 1 \to Y$ is a monomorphism, the terminal object 1 is a pullback of the $\top : 1 \to \Omega$ and $\chi_{eoa} : Y \to \Omega$. Also V is a pullback of the $\top : 1 \to \Omega$ and $\chi_{eoa} \circ e : X \to \Omega$ where $k : V \to X$ is a monomorphism.

$$
\begin{array}{ccc}
1 & \longrightarrow & 1 \\
eoa \downarrow & & \downarrow \top \\
Y & \xrightarrow[\chi_{eoa}]{} & \Omega
\end{array}
$$

$$
\begin{array}{ccccc}
V & \longrightarrow & 1 & \longrightarrow & 1 \\
k \downarrow & & \downarrow eoa & & \downarrow \top \\
X & \xrightarrow[e]{} & Y & \xrightarrow[\chi_{eoa}]{} & \Omega
\end{array}
$$

By hypothesis, for any $\sigma \circ \Omega^e \circ \{\} \circ e \circ a : 1 \to X$ where $\sigma : \Omega^X \to X$ and $\Omega^e \circ \{\} \circ e \circ a : 1 \to \Omega^X$, there exists a morphism $t : 1 \to V$ such that $k \circ t =$

$\sigma \circ \Omega^e \circ \{\} \circ e \circ a$ where $k : V \to X$ is a monomorphism. And by the property of pullback [2],

$$
\begin{array}{ccc}
V & \longrightarrow & 1 \\
k \downarrow & & \downarrow e \circ a \\
X & \xrightarrow{\quad e \quad} & Y
\end{array}
$$

is also a pullback square. Hence it yields

$$e \circ \sigma \circ \Omega^e \circ \{\} \circ e \circ a = e \circ (k \circ t) = (e \circ a \circ !) \circ t = (e \circ a \circ i_1) = e \circ a.$$

Since 1 is a generator and e is an epimorphism, hence $e \circ (\sigma \circ \Omega^e \circ \{\}) = i_Y$.

Theorem 2.3. *Every epimorphism is a retraction if and only if every object is internally projective in a well-pointed topos*

Proof. Let $e : A \to B$ be an epimorphism. Since every object is internally projective, it yields an epimorphism $e^B : A^B \to B^B$. Since

$$
\begin{array}{ccc}
\pi_B(e) & \longrightarrow & A^B \\
\downarrow & & \downarrow e^B \\
1 & \xrightarrow{\quad i_B \quad} & B^B
\end{array}
$$

is a pullback, $k : \pi_B(e) \to 1$ is an epimorphism. By LEMMA 1.5, there exists a morphism $s : 1 \to \pi_B(e)$ such that $k \circ s = i_1$. Since B^* has a right adjoint π_B, $hom(B^*(1), e)$ is natural isomorphic to $hom(1, \pi_B(e))$. Hence, for any $s : 1 \to \pi_B(e)$, there exists a morphism $h : B^*(1) \to e$ such that $e \circ h = B^*(1) = i_B$.

Conversely, let $u : C \to D$ be an epimorphism, then, by hypothesis, there exists a morphism $t : D \to C$ such that $u \circ t = i_D$. We show that $u^X : C^X \to D^X$ is an

epimorphism. Let $m \circ u^X = n \circ u^X$. Then for any $b : X \to D$ and $t : D \to C$, $m \circ u^X(t \circ b) = n \circ u^X(t \circ b)$. It implies that $m(b) = n(b)$ for any $b : X \to D$. Hence for any object X, the functor $(-)^X$ preserves epimorphisms.

Corollary 2.4. *In a well-pointed topos $\mathcal{E}$, the following statements are equivalent:*

(1) Every epimorphism in $\mathcal{E}$ is a retraction.

(2) For any noninitial object A and $f : A \to B$ in $\mathcal{E}$, there exists a morphism $g : B \to A$ such that $f \circ g \circ f = f$.

(3) For any noninitial object A in $\mathcal{E}$, there exists $\sigma : \Omega^A \to A$ such that for all $f : 1 \to \Omega^A$, we have $\sigma \circ f \in f'$ where $f' : A' \to A$ is a monomorphism, provided that $ev \circ (f \times i_A)$ is not the characteristic morphism of $0 \to A$.

(4) Every object of $\mathcal{E}$ is internally projective.

(5) Every epimorphism in $\mathcal{E}$ is locally split.

(6) If $e : A \to B$ is an epimorphism in $\mathcal{E}$, then $\Pi_B(e)$ has global support.

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SANGJI UNIVERSITY, WONJU, 220-702, KOREA.