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SOME CHARACTERIZATIONS OF THE PETTIS INTEGRABILITY VIA FUNCTIONALS

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1. Introduction

Since the invention of the Pettis integral over half century ago, the problem of recognizing the Pettis integrability of a function against an individual condition has been much studied [1,6,7,8,12]. In spite of the R.F. Geitz (1982) and M. Talagrand's (1984) characterization of Pettis integrability, there is often trouble in recognizing when a function is or is not Pettis integrable.

In [1], E. Bator showed that a dual space X^* has the μ-Pettis Integral Property (μ-PIP) with respect to perfect measure μ if and only if for every bounded weakly measurable $f : \Omega \longrightarrow X^*$, $\|w^* - \int_E f d\mu\| = \|D - \int_E f d\mu\|$. In [8] and [10], it is shown how the above statement can be strengthened by dropping the assumption that the measure space must be perfect. The following corollary, proven in [1] for perfect measure, and in general [8], follows easily :

Corollary. *A dual space X^* has the μ - PIP if and only if*

(*) *for every bounded weakly measurable function $f : \Omega \longrightarrow X^*$ and each x^{**} in X^{**}, there exists a bounded sequence (x_n) in X such that $fx_n \longrightarrow x^{**}f$ almost everywhere.*

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In [1], E. Bator asks if the above property $(*)$ ensures Pettis integrability of a given bounded weakly measurable function $f : \Omega \longrightarrow X^*$. The purpose of this paper is to give two characterizations by means of examples and one theorem to show that in general, property $(*)$ does not imply Pettis integrability. The first one is based on well-known example by R. Phillips. The second example is based on [12], and shows that even in the case where X^* is a dual of a separable space, statement $(*)$ fails to imply Pettis integrability.

2. Definitions and Preliminaries

We present some necessary notations and terminology which are needed in our subsequent section. Insofar as possible, we adopt the definitions and notations of [4] and [5]. Throughout this paper, (Ω, Σ, μ) will always be a complete finite measure space, and the dual of a Banach space X will be denoted by X^*.

Definition 2-1. A bounded function $f : \Omega \longrightarrow X$ (resp. $f : \Omega \longrightarrow X^*$ is called *weakly measurable* (resp. *weak* measurable*) if for all x^* in X^* (resp. all x in X) the scalar valued function x^*f (resp. xf) is measurable.

Let $f, g : \Omega \longrightarrow X$ be two weakly measurable functions. They are said to be *weakly equivalent* if for all $x^* \in X^*$, $x^*f = x^*g$ almost everywhere.

Definition 2-2. A weakly measurable function $f : \Omega \longrightarrow X$ is said to be *Dunford integrable* if $x^*f \in L_1(\mu)$ for all $x^* \in X^*$. The Dunford integral of f over $E \in \Sigma$ is defined by the element $x_E^{**} \in X^{**}$ such that $x_E^{**}(x^*) = \int_E x^*f d\mu$ for all $x^* \in X^*$, and denote it by $x_E^{**} = (D) - \int_E f d\mu$.

In the case that $(D) - \int_E f d\mu$ belongs to X for each $E \in \Sigma$, then f is called *Pettis integrable* and we write $(P) - \int_E f d\mu$ instead of $(D) - \int_E f d\mu$ to denote the Pettis integral of f over $E \in \Sigma$.

Example 2-3. *A Dunford integrable function which is not Pettis integrable. Let* $\Omega = [0,1]$ *and* $X = c_0$. *Define* $f : \Omega \longrightarrow X$ *by the equation* $f(t) = (\chi_{(0,1]}(t), 2\chi_{(0,\frac{1}{2}]}(t), \cdots, n\chi_{(0,\frac{1}{n}]}(t), \cdots)$ *for* $t \in [0,1]$. *If* $x^* = (\alpha_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots) \in c_0^* = l_1$, *then* $x^*f = \sum_{n=1}^{\infty} \alpha_n n \chi_{(0,\frac{1}{n}]}$, *a function which is certainly Lebesgue integrable. If* μ *is the Lebesgue measure on* $[0,1]$, *then* $x^*f \in L_1(\mu)$ *for all* $x^* \in X^*$, *i.e.,* f *is Dunford integrable. However, we have*

$$\int_{(0,1]} x^* f d\mu = \sum_{n=1}^{\infty} \alpha_n$$

and the mapping $x^* = (\alpha_n) \longmapsto \sum_{n=1}^{\infty} \alpha_n$ *is the linear functional on* l_1 *corresponding to* $(1,1,\cdots,1,\cdots,) \in l_\infty \setminus c_0$. *Hence,* $(D) - \int_{(0,1]} f d\mu = (1,1,\cdots,1,\cdots,) \notin X$, *so* f *is not Pettis integrable.*

Definition 2-4. The *weak* integral* of $f : \Omega \longrightarrow X^*$ over E, denoted by $(w^*) - \int_E f d\mu$, is the element x_E^* of X^* defined by the equation $x_E^*(x) = \int_E x f d\mu$ for all $x \in X$.

A function $f : \Omega \longrightarrow X^*$ is said to weakly equivalent to zero (resp. weak* equivalent to zero) if for all x^{**} in X^{**}(resp. for all x in X), $x^{**}f = 0$ μ-a.e.(resp. $xf = 0$ μ a.e.).

And a Banach space X is said to have the *μ-Pettis Integral Property*(or μ-PIP) if every bounded weakly measurable function $f : \Omega \longrightarrow X$ is Pettis integrable.

3. The Main Result

The following lemma will be needed in order to ensure Pettis integrability of a given bounded weakly measurable function $f : \Omega \longrightarrow X^*$. For the proof, see [1].

Lemma. *Let (Ω, Σ, μ) be a finite complete measure space. A dual space X^* has the μ-PIP if and only if for every $f : \Omega \longrightarrow X^*$ that is bounded and weakly measurable, $(w^*) - \int_E f d\mu = (D) - \int_E f d\mu$ for every $E \in \Sigma$.*

Example 3-1. *Let w_1 be the first uncountable ordinal, Σ be the σ-algebra of all countable and co-countable subsets of $[0, w_1]$, and $\mu : \Sigma \longrightarrow \{0, 1\}$ be a measure such that*

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is countable,} \\ 1 & \text{if } A^c \text{ is countable.} \end{cases}$$

Define a function $f : [0, w_1] \longrightarrow l_\infty[0, w_1] = (l_1[0, w_1])^*$ by the equation

$$[f(s)](t) = \begin{cases} 0 & \text{if } t < s, \\ 1 & \text{if } t \geq s\ . \end{cases}$$

Claim 1. f is weakly measurable.

The dual of $l_\infty[0, w_1]$ is the space of all bounded and additive measures on $2^{[0,w_1]}$. Fix such a measure β.

There exists a countable subset R of $[0, w_1]$ and a unique decomposition $\beta = \beta_1 + \beta_2$ into bounded additive measures such that for any A, $\beta_1(A) = \beta_1(A \cap R)$ and β_2 vanishes on countable sets. As

$$\beta_1 f(s) = \int_{[0,w_1]} [f(s)](t) \beta_1(t) = \beta_1(R \cap [0, w_1])$$

and

$$\beta_2 f(s) = \int_{[0,w_1]} [f(s)](t) \beta_2(t) = \beta_2([0, w_1]),$$

it follows that $\beta f = \beta_1 f + \beta_2 f = \beta_2([0, w_1])$ μ-a.e..

Claim 2. f is not Pettis integrable.

In fact, the weak*-integral of f is identically zero, but for any $\beta = \beta_1 + \beta_2$, and any set E,

$$\int_E \beta f(s) d\mu(s) = \beta_2([0, w_1])\mu(E).$$

Now, define $\tilde{f} : [0, w_1] \to l_\infty[0, w_1]$ by the equation

$$\tilde{f}(s) = f(s) + \chi_{[0,w_1]}(s).$$

Then $\tilde{f}$ is weakly measurable, not Pettis integrable, but satisfies property $(*)$ of the above, indeed, for any β in the dual of $l_\infty[0, w_1]$, $\beta\tilde{f} = \{2\beta([0, w_1])\}\beta_1\tilde{f}$ where β_1 is any positive norm-one element of $l_1[0, w_1]$.

Remark 1. *The above example shows that any function $f : \Omega \to X^*$ which is weakly measurable and weak* equivalent to zero gives rise to a function satisfying property $(*)$.*

Indeed, when f is such a function and x^{**} is any element of X^{**}, choose a nonzero element z^* in X^* with $x^{**}(z^*) = 0$. Then for any element z in X with $z^*(z) \neq 0$, $\tilde{f}$ is defined by the equation

$$\tilde{f} = f + x^{**} f \frac{z^*}{z^*(z)} \ .$$

Remark 2. *A function $f : \Omega \to X$ defined on a compact Hausdorff space Ω is said to be **universally weakly measurable*** if for every Radon measure μ on Ω, the scalar valued functions x^*f, x^* in X^*, are μ-measurable. If there is a bounded function $f : [0,1] \to l_\infty[0,1]$ such that x^*f is Borel measurable for all x^* in $l_\infty[0,1]^*$, then f is universally weakly measurable. Concerning about the Lebesgue measure on $[0,1]$, f is weak*, but not weakly, equivalent to zero. Hence, by Remark 1, property $(*)$ fails to imply Pettis integrability even in the case where f satisfies the stronger assumption of being weakly universally measurable.

Theorem 3-2. *If a function f with values in $l_\infty(\mathbb{N})$ which satisfies property $(*)$, then f is not Pettis integrable.*

Proof. Let $\Omega = (\{0,1\}^{\mathbb{N}}, \Sigma, \mu)$ be as in [12, Theorem 13-2-1] and let $f : \{0,1\}^{\mathbb{N}} \to l_\infty(\mathbb{N})$ be the function that assigns to each point $a \in \{0,1\}^{\mathbb{N}}$ its characteristic function χ_a.

Write $l_\infty(\mathbb{N})^* = l_1(\mathbb{N}) \oplus c_0^\perp$. In [12, Theorem 13-3-3] it is shown that for any x^* in $c_0^\perp$,

$$x^* f = k_{x^*} \ (= constant) \ \bar{\mu}\text{-a.e.}.$$

Hence, for $x_1^* + x_2^*$ in $l_1(\mathbb{N}) \oplus c_0^\perp$,

$$x^* f = x_1^* f + x_2^* f = x_1^* f + k_{x_2^*} \ \bar{\mu}\text{-a.e.}.$$

If we define a function $\tilde{f} : \{0,1\}^{\mathbb{N}} \to \mathbb{R} \oplus l_\infty(\mathbb{N})$ by the $\tilde{f}(a) = 1 \oplus f(a)$, then for any $k \oplus x^* = k \oplus (x_1^* + x_2^*)$ in $\mathbb{R} \oplus l_\infty(\mathbb{N})^*$,

$$\begin{aligned} \{k \oplus x^*\}\tilde{f} &= k + x^* f \\ &= k + x_1^* f + k_{x_2^*} \quad \bar{\mu}\text{-a.e.} \\ &= \{(k + k_{x_2^*}) \oplus x_1^*\}\tilde{f} \ . \end{aligned}$$

Hence, $\tilde{f}$ satisfies property $(*)$, but is not Pettis integrable since f is not.

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