EXTREMAL DISTANCE AND GREEN'S FUNCTION

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1. Introduction

There are various aspects of the solution of boundary-value problems for second-order linear elliptic equations in two independent variables. One useful method of solving such boundary-value problems for Laplace's equation is by means of suitable integral representations of solutions and these representations are obtained most directly in terms of particular singular solutions, termed Green's functions.

There are no general methods available for finding Green's functions, but for certain geometries the method of extremal length are useful.

In this paper we introduce the concept of extremal distance and investigate its properties. And we will derive some relation between Green's function and a modified form of extremal distance.

2. The properties of extremal distance

DEFINITION 1.1,([7]). *Let D be a domain in the complex plane C and let E_1, E_2 be two sets in the closure of D, and Γ the family of all rectifiable curves in D joining E_1 and E_2. Let P be the set of non-negative real-valued functions $\rho(z)$ on D such that*

$$L_\rho(\gamma) = \int_\gamma \rho |dz|$$

is defined for all $\rho \in P$ and every rectifiable curve γ in D, and

$$A_\rho(D) = \iint_D \rho^2 dxdy \neq 0, \ \infty$$

We consider the finite or positively infinite quantity

$$\lambda_D(E_1, \ E_2) = \sup_{\rho \in P} [\inf_{\gamma \in \Gamma} L_\rho(\gamma)]^2 / A_\rho(D), \tag{1}$$

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which we call the extremal distance between E_1 and E_2 with respect to the domain D.

THEOREM 1.2. *(CONFORMAL INVARIANCE OF EXTREMAL DISTANCE), ([5]). The extremal distance is a conformal invariant of the configuration $(D,\ E_1,\ E_2)$, i.e. if $z^* = f(z)$ is a one-to-one conformal mapping of D upon a domain D^*, taking E_1 to E_1^* and E_2 to E_2^*, then*

$$\lambda_{D^*}(E_1{}^*,\ E_2{}^*) = \lambda_D(E_1,\ E_2).$$

proof. *With $\rho^*(z^*) = \frac{\rho(z)}{|f'(z)|}$ we have*

$$L_{\rho^*}(\gamma^*) = \int_{\gamma^*} \rho^*|dz^*| = \int_{\gamma} \rho|dz| = L_\rho(\gamma),$$

$$\begin{aligned} A_{\rho^*}(D^*) &= \iint_{D^*} (\rho^*)^2 dx^* dy^* = \iint_{D^*} \frac{\rho^2}{|f'(z)|^2} dx^* dy^* \\ &= \iint_D \rho^2 dx dy = A_\rho(D). \end{aligned}$$

The definition of extremal distance remains meaningful if we allow E_1 and E_2 to contain accessible boundary points of D. The following principle is an important tool in dealing with extremal distances.

THEOREM 1.3. *(EXTENSION PRINCIPLE OF EXTREMAL DISTANCE), ([7]). We observe that if $\partial(E_1)$ and $\partial(E_2)$ denote the boundaries of E_1 and E_2, we have*

$$\lambda_D(E_1,\ E_2) = \lambda_D[\partial(E_1),\ \partial(E_2)].$$

proof. *If D' is a domain containing D, and if E_1' and E_2' are subsets of D' containing E_1 and E_2 respectively, then*

$$\lambda_{D'}(E_1',\ E_2') \le \lambda_D(E_1,\ E_2). \tag{2}$$

Because, any $\rho(z)$ defined on D' is also defined on D,

$$A_\rho(D) \le A_\rho(D').$$

And since any curve joining E_1 and E_2 belong to the family Γ' of curves joining E_1' and E_2'

$$inf L_\rho(\gamma) \geq inf L_\rho(\gamma'), \text{ where } \gamma \in \Gamma,\ \gamma' \in \Gamma'. \tag{3}$$

Thus (1) and (3) yield (2).

Now we consider $\partial(E_1)$ and $\partial(E_2)$, then we have $\partial(E_1) \subset E_1$, $\partial(E_2) \subset E_2$; and (2) gives

$$\lambda_D(E_1,\ E_2) \leq \lambda_D[\partial(E_1),\ \partial(E_2)]. \tag{4}$$

On the other hand, since every curve joining E_1 and E_2 will also join their boundaries,

$$\lambda_D(E_1,\ E_2) \geq \lambda_D[\partial(E_1),\ \partial(E_2)]. \tag{5}$$

Combining (4) and (5) we have

$$\lambda_D(E_1,\ E_2) = \lambda_D[\partial(E_1),\ \partial(E_2)].$$

2. Some relation between Green's function and extremal distance

Every doubly connected domain is conformally equivalent to an annulus.

LEMMA 2.1. *([3]). Every doubly connected domain B can be mapped conformally onto an annulus $\{z | r_1 < |z| < r_2\}$.*

Such a mapping is called a canonical mapping and the corresponding annulus a canonical image of B.

LEMMA 2.2. *If A_1 and A_2 denote the two components of the boundary of an annulus $\{z | r_1 \leq |z| \leq r_2\}$, then*

$$\lambda_A(A_1,\ A_2) = \frac{1}{2\pi} \log \frac{r_2}{r_1}. \tag{6}$$

proof. *Let Γ be the family of all rectifiable curves γ in A which joining A_1 and A_2. We let*

$$L_\rho(\Gamma) = \inf_{\gamma \in \Gamma} L_\rho(\gamma)$$

For any ρ, we have

$$\int_{r_1}^{r_2} \rho dr \geq L_\rho(\Gamma),$$

$$\iint_A \rho dr d\theta = \int_0^{2\pi} \int_{r_1}^{r_2} \rho dr d\theta \geq 2\pi L_\rho(\Gamma),$$

$$\begin{aligned} 4\pi^2[\mathrm{L}_\rho(\Gamma)]^2 &\leq [\iint_A \rho dr d\theta]^2 \\ &\leq [\int_0^{2\pi} \int_{r_1}^{r_2} \rho^2 \cdot \frac{1}{r} dr d\theta] \cdot [\iint_A r dr d\theta] \\ &= 2\pi \log \frac{r_2}{r_1} \cdot \iint_A \rho^2 r dr d\theta, \\ \frac{[L_\rho(\Gamma)]^2}{A_\rho(A)} &\leq \frac{1}{2\pi} \log \frac{r_2}{r_1}. \end{aligned}$$

Thus

$$\lambda_A(A_1,\ A_2) \leq \frac{1}{2\pi} \log \frac{r_2}{r_1}. \tag{7}$$

On the other hand for $\rho = 1/r$, we have

$$L_{\frac{1}{r}}(\Gamma) = \log \frac{r_2}{r_1}, \quad A_{\frac{1}{r}}(A) = 2\pi \log \frac{r_2}{r_1}.$$

Thus

$$\lambda_A(A_1,\ A_2) \geq \frac{1}{2\pi} \log \frac{r_2}{r_1}. \tag{8}$$

Therefore by virtue of (7), (8), we obtain (6).

DEFINITION 2.3. *([4]). Two points P_1 and P_2 are called inverse points with respect to a circle of radius R and center P if P, P_1, P_2 lie, in that order, on the same straight line and if the distances $\overline{PP_1}$ and $\overline{PP_2}$ are related by*

$$\overline{PP_1} \cdot \overline{PP_1} = R^2.$$

Now we are ready to state our result.

THEOREM 2.4. *Let z_1 and z_2 be two interior points of $C_a = \{z|\ |z| = a\}$ and z_1' be the inverse point of z_1 with respect to C_a. And let $G(z_1,\ z_2)$ be the Green's function which vanishes on C_a for Laplace's equation within C_a.*

If E is a doubly connected domain whose boundary consists of two components $E_1,\ E_2$ and canonical image E^* of E is $E^* = \{z|0 < r_1r_3 < |z| < ar_2 < \infty\}$, then

$$G(z_1,\ z_2) = \lambda_E(E_1,\ E_2)$$

where $r_1 = |z_1|$, $r_2 = |z_1 - z_2|$ and $r_3 = |z_1' - z_2|$.

Proof. *It is well known that the Green's function is*

$$G(z_1,\ z_2) = \frac{1}{2\pi}\log\frac{ar_2}{r_1r_3}, \quad ([6]).$$

Therefore by virtue of theorem 1.2, theorem 1.3, Lemma 2.1 and Lemma 2.2,

$$G(z_1,\ z_2) = \lambda_E * (C_{r_1r_3},\ C_{ar_2}) = \lambda_E(E_1,\ E_2).$$

This completes the proof of the theorem.

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