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A NOTE ON QUASI-OPEN MAPS

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ABSTRACT. Let $f : X \to Y$ be quasi-open. We show that: (1) If $A \subset X$ is open, $f|A$ is quasi-open, (2) $f : X \to f(X)$ is quasi-open. (3) And let $f_\alpha : X_\alpha \to Y_\alpha$ be quasi-open. Then $\Pi f_\alpha : \Pi X_\alpha \to \Pi Y_\alpha$, defined by $\{x_\alpha\} \to \{f_\alpha(x_\alpha)\}$, is quasi-open. (4) Lastly, if $f_i : X_i \to Y$ are quasi-open, $i = 1, 2$, then $F : X_1 \oplus X_2 \to Y$, defined by $F(x) = f_i(x)$, $x \in X_i$, is also quasi-open.

1. Introduction

The concept of a quasi-open map was introduced by Kao[3] in 1983. Some characterizations of M_1-spaces, in terms of quasi-open maps, have been given by Kao[3].

The continuous maps and the quasi-open maps are not related. See the Examples of this note. But the quasi-open maps have the properties which are similar to those of the continuous maps. The purpose of this note is to derive the characterizations of quasi-open maps

Let X, Y and Z be topological spaces with no separation axioms assumed unless explicitly stated.

The interior of a subset U of X will be denoted by $\mathrm{Int}(U)$. Notations and terminologies not explained here but used in this note are taken from Dugundji[2].

2. Results

Definition 1 [3]. A mapping $f : X \to Y$ is called quasi-open if $\mathrm{Int}(f(U)) \neq \emptyset$ for every non-empty open subset $U \subset X$.

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Example 1. *Let* $X = \{a, b, c\}$, *and* $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ *be a topology on* X. *Let* $Y = \{p, q\}$, *and* $\sigma = \{\emptyset, \{p\}, Y\}$ *be a topology on* Y. *Define* $f : (X, \tau) \to (Y, \sigma)$ *by* $f(a) = p$ *and* $f(b) = f(c) = q$. *Then* f *is continuous but not quasi-open.*

Example 2. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ *be a topology on* X. *Let* $Y = \{p, q, r\}$ *and* $\sigma = \{\emptyset, \{p\}, \{r\}, \{p, r\}, \{q, r\}, Y\}$ *be a topology on* Y. *Define* $g : (X, \tau) \to (Y, \sigma)$ *by* $g(a) = p$, $g(b) = q$, $g(c) = r$. *Then* g *is quasi-open but not continuous.*

Lemma 1 [3]. *If* $f : X \to Y$ *is open,* f *is quasi-open.*

Lemma 2(cf. [1]). *If* $f : X \to Y$ *is local homeomorphism,* f *is quasi-open.*

Proof. Every local homeomorphism is a countinuous open map[1]. By Lemma 1, f is quasi-open.

Let ΠX_α be the product space with the product topology. Then the β-th projection map $\pi_\beta : \Pi X_\alpha \to X_\beta$ is continuous, open and surjective. Hence, by Lemma 1, we obtain the following lemma.

Lemma 3 [3]. π_β *is quasi-open.*

Lemma 4 [Composition](cf. [3]). *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be quasi-open maps. Then* $g \circ f$ *is quasi-open.*

Proof. Let U be any non-empty open set in X. Since f is quasi-open, $\mathrm{Int}(f(U)) \neq \emptyset$. Since g is also quasi-open, $\mathrm{Int}(g(\mathrm{Int}(f(U))) \neq \emptyset$.

But we know that $\mathrm{Int}(g(\mathrm{Int}(f(U))) \subset \mathrm{Int}(g(f(U)))$. Hence $\mathrm{Int}(g(f(U))) \neq \emptyset$. This completes the proof.

If A is an open subset of X, then the inclusion $i : A \to X$ is open [2].

Proposition 5 [Restriction of Domain]. *Let* $f : X \to Y$ *be a quasi-open map, and* A *open subspace of* X. *Then* $f|A : A \to Y$ *is quasi-open.*

Proof. Let U be any non-empty open set in A. Then U is a non-empty open set in X. We know that $f|A = f \circ i$, where $i : A \to X$ is an inclusion, and that i is open [2]. By Lemma 4, we get the result.

Proposition 6 [Restriction of Range]. *If* $f : X \to Y$ *is quasi-open and* $f(X)$ *is taken the subspace topology, then* $f : X \to f(X)$ *is quasi-open.*

Proof. Let U be any non-empty open set in X. Then $\mathrm{Int}_{f(X)}(f(U)) \supset \mathrm{Int}_Y(f(U)) \cap f(X) = \mathrm{Int}_Y(f(U))$. Since f is quasi-open, $\mathrm{Int}_Y(f(U)) \neq \emptyset$. Hence $f : X \to f(X)$ is quasi-open.

Proposition 7. *Let $f_\alpha : X_\alpha \to Y_\alpha$ be onto for each α. Define $\Pi f_\alpha : \Pi X_\alpha \to \Pi Y_\alpha$ by $\{x_\alpha\} \to \{f_\alpha(x_\alpha)\}$. If f_α is quasi-open for each α, Πf_α is quasi-open.*

Proof. Let $U = U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \Pi_{\alpha \neq \alpha_i} X_\alpha$ be a non-empty basic open set in ΠX_α. Then $\mathrm{Int}(\Pi f_\alpha(U)) = \mathrm{Int}(f_{\alpha_1}(U_{\alpha_1})) \times \cdots \times \mathrm{Int}(f_{\alpha_n}(U_{\alpha_n})) \times \Pi_{\alpha \neq \alpha_i} Y_\alpha$ is non-empty. Hence Πf_α is quasi-open.

Let $X_1 \oplus X_2$ be a sum of disjoint topological spaces X_1 and X_2. Define $F : X_1 \oplus X_2 \to Y$ by $F(x) = f_i(x)$ if $x \in X_i$, where $f_i : X_i \to Y, i = 1, 2$.

Proposition 8. *If $f_i : X_i \to Y$ are quasi-open, $i = 1, 2, F$ is quasi-open.*

Proof. Let U be any non-empty open set in $X_1 \oplus X_2$. Then $U \cap X_i$ are open in X_i by the definition of a topological sum. Since f_i are quasi-open, $\emptyset \neq \mathrm{Int}(f_1(U \cap X_1)) \cup \mathrm{Int}(f_2(U \cap X_2)) \subset \mathrm{Int}(f_1(U \cap X_1) \cup f_2(U \cap X_2)) = \mathrm{Int}(F(U))$. Hence F is quasi-open.

Corollary 9. *If $f_i : X_i \to Y$ are quasi-open for $i = 1, 2, \cdots, n$, the map $F : \oplus_{i=1}^n X_i \to Y$, defined by $F(x) = f_i(x)$ if $x \in X_i$, is quasi-open.*

References

1. N. Bourbaki, *General Topology, Part 1*, Addison-Wesly, Reading, Mass., 1966.
2. J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.
3. K. S. Kao, *A Note on M_1-spaces*, Pacific J. Math. **108(1)** (1983), 121-128.

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