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ON p-HYPONORMAL OPERATORS ON A HILBERT SPACE

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ABSTRACT. Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the $*$-algebra of all bounded linear operators on $\mathcal{H}$. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be p-hyponormal if $(T^*T)^p - (TT^*)^p \geq 0$ for $0 < p < 1$. If $p = 1$, T is hyponormal and if $p = \frac{1}{2}$, T is semi-hyponormal. In this paper, by using a technique introduced by S. K. Berberian, we show that the approximate point spectrum $\sigma_{ap}(T)$ of a pure p-hyponormal operator T is empty, and obtains the compact perturbation of T.

1. Introduction

Throughout this paper, the letter $\mathcal{H}$ is used for a complex separable Hilbert space and the $*$-algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be $p-$ *hyponormal* if $(T^*T)^p - (TT^*)^p \geq 0$ for $0 < p < 1$. If $p = 1$, T is hyponormal and if $p = \frac{1}{2}$, T is semi-hyponormal. It is well known that a p-hyponormal operator is q-hyponormal for $q \leq p$ ([1]). But the converse of the above statement is not true in general.

Hyponormal operators have been studied by many authors. The semi-hyponormal operator was first introduced by D. Xia in ([7]). The p-hyponormal operators have been studied by A. Aluthge in ([1]). All those classes are related as the following inclusions ;

$$\text{Hyponormal} \subset \text{Semi-hyponormal} \subset p-\text{hyponormal for } 0 < p < \frac{1}{2}.$$

$$\text{Hyponormal} \subset p-\text{hyponormal for } \frac{1}{2} < p < 1 \subset \text{Semi-hyponormal}.$$

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Moreover, those inclusions are known to be proper ([1],[5],[8]).

Let $U|T|$ be the polar decomposition of T, where U is a partial isometry, $|T|$ is a positive square root of T^*T and $\ker T = \ker|T| = \ker U$.

Proposition 1.1 ([1],[5]). *Let $T = U|T|$ be p-hyponormal. Then the operator $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is hyponormal for $\frac{1}{2} \le p < 1$, and $(p+\frac{1}{2})$- hyponormal for $0 < p < \frac{1}{2}$.*

Proposition 1.2 ([5]). *Let $T = U|T|$ be p-hyponormal. If $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is normal, then $\tilde{T} = T$.*

In Section 2, we introduce the technique of S. K. Berberian ([2]). For a p-hyponormal T and a unital faithful $*$-representation ϕ which was obtained by S. K. Berberian, we investigate the relation between T and $\phi(T)$.

In Section 3, using the technique introduced by S. K. Berberian, we show that the approximate point spectrum $\sigma_{ap}(T)$ of a p-hyponormal operator T for $0 < p < \frac{1}{2}$ is empty, and obtain the compact perturbation of T.

2. A pure operator

The spectrum, the point spectrum and the approximate point spectrum of an operator T in $\mathcal{L}(\mathcal{H})$ are denoted by $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$, respectively.

In [2], S. K. Berberian constructed an extension $\mathcal{K}$ of $\mathcal{H}$ by means of bounded sequences in $\mathcal{H}$ and the Banach limits, and obtained the faithful $*$-representation ϕ of $\mathcal{L}(\mathcal{H})$ on $\mathcal{K}$. Here we present this technique in a simplified form.

Theorem 2.1 ([2]). *Let $\mathcal{H}$ be a separable complex Hilbert space. Then there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a unital faithful $*$-representation ϕ of $\mathcal{L}(\mathcal{H})$ on $\mathcal{K}$ with the following properties :*

(1) $\|\phi(T)\| = \|T\|$ *and* $\phi(A) \le \phi(B)$ *whenever* $A \le B$

(2) $\sigma(T) = \sigma(\phi(T))$ *and* $\sigma_{ap}(T) = \sigma_{ap}(\phi(T)) = \sigma_p(\phi(T))$

An operator is said to be *reducible* if it has a nontrivial reducing subspace. If an operator is not reducible, then it is called *irreducible*.

Proposition 2.2. *If T is an irreducible operator, then $\phi(T)$ is an irreducible operator.*

Proof. Suppose that $\phi(T)$ is reducible. Then there exists a nontrivial subspace $\mathcal{M}$ of $\mathcal{K}$ such that $\phi(T)\mathcal{M} \subset \mathcal{M}$ and $\phi(T^*)\mathcal{M} \subset \mathcal{M}$, that is, $\phi(T)\{x_n\}' = \{Tx_n\}' \in \mathcal{M}$ and $\phi(T)^*\{x_n\}' = \{T^*x_n\}' \in \mathcal{M}$ for all $\{x_n\}' = \{x_n\} + \mathcal{N} \in \mathcal{M}$, where $\mathcal{N} = \{\{x_n\} | x_n \in \mathcal{H}, LIM\{\|x_n\|\} = 0\}$, LIM means the Banach limit. Put $\mathcal{M}_1 = \{x_n | \{x_n\}' \in \mathcal{M}\}$. If $\mathcal{M}_1 = \mathcal{H}$, then obviously $\mathcal{M} = \mathcal{K}$ by the construction of $\mathcal{K}$, which contradicts that $\phi(T)$ is reducible. Thus, $\mathcal{M}_1$ is a proper subspace of $\mathcal{H}$. Hence for all $x_n \in \mathcal{M}$, Tx_n and T^*x_n are in $\mathcal{M}_1$, which contradicts that T is irreducible. Therefore, $\phi(T)$ is an irreducible operator.

An operator T is *pure* if it has no reducing subspace on which it is normal.

Proposition 2.3. *If T is a pure operator, then $\phi(T)$ is a pure operator.*

Proof. If $\phi(T)$ is not pure, then there exists a proper subspace $\mathcal{M}$ of $\mathcal{K}$ such that the restriction of $\phi(T)$ to $\mathcal{M}$ is normal. Put $\mathcal{M}_1 = \{x_n | \{x_n\}' \in \mathcal{M}\}$. Since ϕ is a unital faithful $*$-representation of $\mathcal{L}(\mathcal{H})$ on $\mathcal{K}$, the restriction of T to $\mathcal{M}_1$ is normal, which contradicts that T is pure.

Proposition 2.4. *If an operator T is pure p-hyponormal, then $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is a pure operator.*

Proof. Suppose that $\tilde{T}$ is not pure. Then there is a subspace $\mathcal{H}_1$ of $\mathcal{H}$ such that $\tilde{T}|_{\mathcal{H}_1}$ is normal. By Proposition 1.2, $\tilde{T}|_{\mathcal{H}_1} = T|_{\mathcal{H}_1}$, which contradicts that T is pure.

Proposition 2.5. *If $T \in \mathcal{L}(\mathcal{H})$, then $\phi(|T|) = |\phi(T)|$ and $\phi(|T|_l) = |\phi(T)|_l$, where $|T|_l$ is a positive square root of TT^*.*

Proof. It is clear that $|\phi(T)|^2 = \phi(T)^*\phi(T) = \phi(T^*T) = \phi(|T|^2) = \phi(|T|)^2$. By the uniqueness of the square root of a positive operator, we have $\phi(|T|) = |\phi(T)|$. Similarly, $\phi(|T|_l) = |\phi(T)|_l$.

Proposition 2.6. *If T is a p-hyponormal operator for $0 < p < \frac{1}{2}$, then $\phi(T)$ is a p-hyponormal operator.*

Proof. We need only to prove for $p = \frac{1}{2^n}$ for some n. By the assumption, we have

$$(T^*T)^{\frac{1}{2^n}} - (TT^*)^{\frac{1}{2^n}} \geq 0.$$

Thus, by Proposition 2.5,

$$\begin{aligned}&(\phi(T^*)\phi(T))^{\frac{1}{2^n}} - (\phi(T)\phi(T^*))^{\frac{1}{2^n}}\\ =&\phi((T^*T))^{\frac{1}{2^n}} - \phi((TT^*))^{\frac{1}{2^n}}\\ =&\phi((T^*T)^{\frac{1}{2^n}}) - \phi((TT^*)^{\frac{1}{2^n}})\\ =&\phi((T^*T)^{\frac{1}{2^n}} - (TT^*)^{\frac{1}{2^n}}) \geq 0.\end{aligned}$$

3. Spectra of p-hyponormal operators

For every operator $T \in \mathcal{L}(\mathcal{H})$, there is a Cartesian decomposition $T = X + iY$, where $X = \frac{1}{2}(T + T^*)$ and $Y = \frac{1}{2i}(T - T^*)$. The operators X and Y are called the *real* and *imaginary* parts of T, respectively.

The joint point spectrum $\sigma_{jp}(T)$ of $T = X + iY$ is the set of all complex numbers $z = x + iy$ (x and y are real numbers) such that there exists a common nonzero eigenvector f of X and Y such that

$$Xf = xf \quad \text{and} \quad Yf = yf.$$

In addition, $z = re^{i\theta} \in \sigma_{jp}(T)$ if and only if there exists a nonzero vector f such that

$$Tf = zf \quad \text{and} \quad T^*f = \bar{z}f,$$

equivalently, there exists a nonzero vector f such that

$$Uf = e^{i\theta}f \quad \text{and} \quad |T|_r f = rf \text{ ([8])}.$$

Proposition 3.1 ([5]). *If an operator $T = U|T|$ is p-hyponormal for $0 < p < \frac{1}{2}$, then $\sigma_{jp}(T) = \sigma_p(T)$.*

Proposition 3.2. *If an operator T is pure p-hyponormal for $0 < p < \frac{1}{2}$, then $\sigma_p(T) = \emptyset$.*

Proof. Suppose that there exists a $\lambda \in \sigma_p(T)$. Since $\sigma_{jp}(T) = \sigma_p(T)$, by Proposition 3.1, there is a nonzero vector f such that $Tf = \lambda f$ and $T^*f = \bar{\lambda}f$. Thus, the one-dimensional vector space

$$\mathcal{M} = \{\mu f | \mu \in \mathbb{C}\}$$

reduces T, and $TT^*f = T\bar{\lambda}f = \bar{\lambda}Tf = |\lambda|^2 f$ and $T^*T(f) = T^*(Tf) = \lambda T^* f = |\lambda|^2 f$. Hence, the restriction of T to $\mathcal{M}$ is normal, which contradicts the assumption that T is pure. Therefore, $\sigma_p(T) = \emptyset$.

Proposition 3.3. *If an operator T is pure p-hyponormal for $0 < p < \frac{1}{2}$, then $\sigma_{ap}(T) = \emptyset$.*

Proof. Suppose that T is a pure p-hyponormal operator for $0 < p < \frac{1}{2}$. Then by Proposition 2.3 and 2.6, $\phi(T)$ is a pure p-hyponormal operator. Thus, $\sigma_p(\phi(T)) = \emptyset$. By Proposition 2.1, since

$$\sigma_{ap}(T) = \sigma_{ap}(\phi(T)) = \sigma_p(\phi(T)), \ \sigma_{ap}(T) = \emptyset.$$

By Proposition 1.1 and 2.5, we have the following.

Corollary 3.4. *Let $T = U|T|$ be pure p-hyponormal. If $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, then $\sigma_{ap}(\tilde{T}) = \emptyset$.*

The joint approximate point spectrum, $\sigma_{ja}(T)$ of an operator $T = X + iY$ is the set of all complex numbers $z = x + iy$ which there exists a sequence $\{f_n\}$ of unit vectors such that

$$\lim_{n\to\infty} \|(X - xI)f_n\| = \lim_{n\to\infty} \|(Y - yI)f_n\| = 0.$$

It is evident that $z \in \sigma_{ja}(T)$ if and only if there exists a sequence $\{f_n\}$ of unit vectors such that

$$\lim_{n\to\infty} \|(T - zI)f_n\| = \lim_{n\to\infty} \|(T^* - \bar{z}I)f_n\| = 0.$$

It is also evident that $\sigma_{ja}(T) \subset \sigma_{ap}(T)$ for all $T \in \mathcal{L}(\mathcal{H})$, and moreover, for a normal operator T, we have

$$\sigma_{ja}(T) = \sigma_{ap}(T) = \sigma(T) \quad ([8]).$$

Proposition 3.5 ([3])**.** *If $T = U|T|$ is a p-hyponormal operator, then $\sigma_{ja}(T) = \sigma_{ap}(T)$.*

An operator T in $\mathcal{L}(\mathcal{H})$ is said to be *strongly normal* at λ if there exists an orthonormal sequence $\{x_n\}$ in $\mathcal{H}$ such that

$$\lim_{n\to\infty} \{\|(T - \lambda I)x_n\| + \|(T - \lambda I)^* x_n\|\} = 0$$

An operator T is said to be *strongly normal* on a subset M of $\mathbb{C}$ if T is strongly normal at every point of M.

Corollary 3.6. *If $T = U|T|$ is a p-hyponormal operator, then T is strongly normal on $\sigma_{ap}(T)$.*

Proof. By Proposition 3.5, it is clear.

Proposition 3.7 ([6]). *Let $\mathcal{K}(\mathcal{H})$ be the ideal of compact operators acting on $\mathcal{H}$ and let $T \in \mathcal{L}(\mathcal{H})$. Then T is strongly normal on a subset M of $\mathbb{C}$ if and only if for every diagonal operator D with spectrum in M there exists a compact operator $K \in \mathcal{K}(\mathcal{H})$ with trace norm, $E \in \mathcal{L}(\mathcal{H})$ and an isometric operator U of $\mathcal{H}$ onto $\mathcal{H} \oplus \mathcal{H}$ such that*

$$T = U^{-1}(D \oplus E)U + K.$$

By Corollary 3.6 and Proposition 3.7, we have the following result :

Proposition 3.8. *Let D be a diagonal operator in $\mathcal{L}(\mathcal{H})$. If T is p-hyponormal, $\sigma(D) \subset \sigma_{ap}(T)$ and given $\epsilon > 0$, then we may write*

$$T = (D \oplus E) + K,$$

where $K \in \mathcal{K}(\mathcal{H})$ with $\|K\|_1 < \epsilon$ and $E \in \mathcal{L}(\mathcal{H})$, and $\|K\|_1$ is the trace norm of K in $\mathcal{K}(\mathcal{H})$.

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