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# A CERTAIN SUBGROUP OF THE WEYL GROUP OF SOME KAC-MOODY ALGEBRAS

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ABSTRACT. In this paper, we construct the minimal set of generators which generate the subgroup $T$ of the Weyl group of Kac-Moody algebra.

## 1. Notation and some basic facts about root systems of Kac-Moody algebras

We first recall some of the basic definitions in Kac-Moody theory.

An $n \times n$ integral matrix $A = (a_{ij})_{i,j=1}^{n}$ is called a generalized Cartan matrix(GCM) if

$$\begin{cases} a_{ii} = 2, & i = 1, 2, \dots, n, \\ a_{ij} \leq 0 & \text{if } \ i \neq j, \\ a_{ij} = 0 & \text{implies } \ a_{ji} = 0. \end{cases} \tag{1.1}$$

A realization of $A$ is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a complex vector space, $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ are indexed subsets in $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively, satisfying the following three conditions;

$$\begin{cases} \Pi \text{ and } \Pi^\vee \text{ are linearly independent} \\ \alpha_j(\alpha_i^\vee) = a_{ij} \quad (i, j = 1, 2, \dots, n) \\ \dim \mathfrak{h} = 2n - l, \quad \text{where } l = \operatorname{rank} A. \end{cases} \tag{1.2}$$

An $n \times n$ matrix $A = (a_{ij})_{i,j=1}^{n}$ is called symmetrizable if there exists an invertible diagonal matrix $D$ and a symmetrix matrix $B = (b_{i,j})$ such that $DA = B$.

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The Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ with the generalized Cartan matrix $A$ is the Lie algebra generated by the elements $e_i$, $f_i$ $(i = 1, 2, \dots, n)$ and $\mathfrak{h}$ with the following defining relations;

$$\begin{cases} [h, h'] = 0 \quad \text{for } h,\ h' \in \mathfrak{h}, \\ [h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i \quad (i = 1, 2, \dots, n;\ h \in \mathfrak{h}), \\ [e_i, f_i] = \delta_{ij}\alpha_i^\vee \quad \text{for } i, j = 1, 2, \dots, n, \\ (ad\, e_i)^{1-a_{ij}}(e_j) = (ad\, f_i)^{1-a_{ij}}(f_j) = 0 \quad \text{for } i \neq j. \end{cases} \tag{1.3}$$

The elements of $\Pi$ (resp. $\Pi^\vee$) are called the simple roots (resp. simple coroots) of $\mathfrak{g}$.

For each $i \in \{1, 2, \dots, n\}$, let $r_i \in \mathrm{Aut}(\mathfrak{h}^*)$ be the simple reflection on $\mathfrak{h}^*$ defined by

$$r_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i.$$

The subgroup $W$ of $GL(\mathfrak{h}^*)$ generated by the $r_i$'s $(i = 1, 2, \dots, n)$ is called the Weyl group of $\mathfrak{g}$.

We adopt the following notation: for a real column vector ${}^t(u_1, u_2, \dots, u_n)$, we write $u > 0$ if all $u_i > 0$ and $u \geq 0$ if all $u_i \geq 0$.

**Theorem 1.1.** [1] *Let $A$ be a real $n \times n$ generalized Cartan matrix. Then one and only one possibilities holds for both $A$ and ${}^tA$:*

(Fin) $\det A \neq 0$; *there exists $u > 0$ such that $Au > 0$; $Av \geq 0$ implies $v > 0$ or $v = 0$.*

(Aff) $\mathrm{corank}\, A = 1$; *there exists $u > 0$ such that $Au = 0$; $Av \geq 0$ implies $Av = 0$.*

(Ind) *there exists $u > 0$ such that $Au < 0$; $Av \geq 0$, $v \geq 0$ implies $v = 0$.*

Referring to cases (Fin), (Aff) or (Ind), we will say that $A$ is of finite, affine or indefinite type, respectively.

Let $A = (a_{ij})_{i,j=1}^n$ be a generalized Cartan matrix. We associate to $A$ a graph $S(A)$, called the Dynkin diagram of $A$ as follows. If $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, then the vertices $i$ and $j$ are connected by $|a_{ij}|$ lines, and these lines are equipped with an arrow pointing toward $i$ if $|a_{ij}| > 1$. If $a_{ij}a_{ji} > 4$, the vertices $i$ and $j$ are connected by a bold-faced line equipped with an ordered pair of integers $(|a_{ij}|, |a_{ji}|)$.

An indecomposable generalized Cartan matrix $A$ is said to be of strictly hyperbolic type(resp. hyperbolic type) if it is of indefinite type and connected proper subdiagram of $S(A)$ is of finite(resp. finite or affine) type.

Suppose that $A$ is symmetrizable generalized Cartan matrix. Then the non-degenerate symmetric bilinear form ( , ) can be defined on $\mathfrak{h}^*$ and $A$ can be expressed as

$$A = \left( \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \right)_{i,j=1}^{n}$$

which is the same as the usual expression of the generalized Cartan matrix[3].

## 2. Structure of the Weyl group of some Kac-Moody algebra

We know the Weyl group $W$ is a Coxter group generated by $r_i, \ldots, r_n$ and satisfies the following relations

$$r_i^2 = 1 \quad (r_i r_j)^{m_{ij}} = 1 \ \ (i \neq j)$$

where $m_{ij} \in [2, \infty)$ are given in terms of the generalized Cartan matrix by following table;

| $a_{ij}\, a_{ij}$ | 0 | 1 | 2 | 3 | $\geq 4$ |
|---|---|---|---|---|---|
| $m_{ij}$ | 2 | 3 | 4 | 6 | $\infty$ |

**Definition 2.1** A Coxter group generated by $\{r_i | i \in I\}$ is called a free Coxter group, if the order of $r_i r_j$ is infinite for all $i \neq j \in I$.

**Lemma 2.2.** *If $(\alpha_i, \alpha_i)(\alpha_j, \alpha_j) \leq (\alpha_i, \alpha_j)^2$ for $i, j = 1, \ldots, n$, then $W$ is a free Coxter group generated by $r_1, \ldots, r_n$.*

*Proof.* If $i \neq j$, then

$$\begin{aligned} a_{ij} a_{ji} &= \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \frac{2(\alpha_j, \alpha_i)}{(\alpha_j, \alpha_j)} \\ &= \frac{4(\alpha_i, \alpha_j)^2}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)} \\ &\geq \frac{4(\alpha_i, \alpha_j)^2}{(\alpha_i, \alpha_j)^2} = 4. \end{aligned}$$

The above table shows $r_i r_j$ has infinite order.

From now on, we always assume that $A = (a_{ij})_{i,j=0}^{n}$ is an $(n+1) \times (n+1)$ indecomposable symmetrizable generalized Cartan matrix, and $S(A)$ is the Dynkin diagram corresponding to $A$. Let $W = \langle r_0, r_1, \dots, r_n \rangle$ be the Weyl group of $A$. Denote $\mathring{W} = \langle r_1, \dots, r_n \rangle$. Set $T = \{r_{\beta\alpha_0} \mid \beta \in \mathring{W}\}$.

Recall that for each real root $\alpha$ we have defined a reflection $r_\alpha$ by

$$r_\alpha(\lambda) = \lambda - \lambda(\alpha^\vee)\alpha \quad (\lambda \in \mathfrak{h}^*).$$

Then $r_{\beta\alpha_0} = \beta r_0 \beta^{-1}$.

In this paper, we shall normalize $(,)$ so that $(\alpha_0, \alpha_0) = 1$.

**Lemma 2.3.** [4] *Let $r_{i_1} r_{i_2} \cdots r_{i_s} = 1$, $r_{i_j} r_{i_{j+1}} \neq r_{i_{j+1}} r_{i_j}$, where $s \geq 2$ and $s$ is minimal for such expressions. Then $s = 2m \geq 4$ and $r_{i_1} = r_{i_3} = \cdots = r_{i_{2m-1}}$, $r_{i_2} = r_{i_4} = \cdots = r_{i_{2m}}$. Furthermore, $m = 3, 4$ or $6$.*

**Theorem 2.4.** *Let $P = \{\ i \mid (\alpha_i, \alpha_i) \leq (\alpha_0, \alpha_i)^2\}$ and $Q = \{\ i \mid (\alpha_0, \alpha_i) = 0\}$. If $P \cup Q = \{0, 1, \dots, n\}$, then there exists a minimal subset $I$ of $\mathring{W}$ such that $\langle r_{\beta\alpha_0} | \beta \in I \rangle = T$*

*Proof.* Set $\widetilde{\mathring{W}} = \{\omega \in \mathring{W} \mid \omega\alpha_0 = \alpha_0\}$. Clearly $\widetilde{\mathring{W}} = \langle r_i \mid r_i\alpha_0 = \alpha_0 \rangle$ and $\widetilde{\mathring{W}}$ is a subgroup of $\mathring{W}$. Construct a set $I$ by choosing exactly one element from each left coset of $\mathring{W}/\widetilde{\mathring{W}}$.

First, we show that $\langle r_{\beta\alpha_0} \mid \beta \in \mathring{W} \rangle \subset \langle r_{\beta\alpha_0} \mid \beta \in I \rangle$. By the construction, for each $\omega \in \mathring{W}$, there exists only one $\omega' \in I$ such that $\omega\widetilde{\mathring{W}} = \omega'\widetilde{\mathring{W}}$. This implies $\omega^{-1}\omega' \in \widetilde{\mathring{W}}$ and hence $\omega\alpha_0 = \omega'\alpha_0$. Therefore $r_{\omega\alpha_0} = r_{\omega'\alpha_0}$. Next, we shall show that $I$ has no proper subset $J$ such that $\langle r_{\beta\alpha_0} \mid \beta \in J \rangle = T$. Suppose $J \subsetneq I$ and $\langle r_{\beta\alpha_0} \mid \beta \in I \rangle = T$.

Then there exists $\beta_0 \in I$ with $\beta_0 \notin J$. Since $\langle r_{\beta\alpha_0} \mid \beta \in J \rangle = T$, there exist $\beta_1, \dots, \beta_t$ such that $r_{\beta_0\alpha_0} = r_{\beta_1\alpha_0} r_{\beta_2\alpha_0} \cdots r_{\beta_t\alpha_0}$ with $t$ minimal. Then we have

$$r_0\beta_1^{-1}\beta_2 r_0 \cdots \beta_t r_0 \beta_t^{-1}\beta_0 r_0 \beta_0^{-1}\beta_1 = 1.$$

We claim that $\beta_i^{-1}\beta_{i+1} r_0 \neq r_0 \beta_i^{-1}\beta_{i+1}$ for all $1 \leq i \leq t-1$ and $\beta_t^{-1}\beta_0 r_0 \neq r_0\beta_t^{-1}\beta_0$. Suppose not then $\beta_{i+1} r_0 \beta_{i+1}^{-1} = \beta_i r_0 \beta_i^{-1}$ for some $i$, and hence $r_{\beta_{i+1}\alpha_0} = r_{\beta_i\alpha_0}$, which contradicts to the minimality of $t$. Similarly, suppose that $\beta_t^{-1}\beta_0 r_0 = r_0\beta_t^{-1}\beta_0$. Then $\beta_t^{-1}\beta_0 r_0 \alpha_0 = r_0 \beta_t^{-1}\beta_0\alpha_0$.

This implies $-\beta_t^{-1}\beta_0\alpha_0 = r_0\beta_t^{-1}\beta_0\alpha_0$, and hence $\beta_t^{-1}\beta_0\alpha_0 = \alpha_0$. This contradicts to the fact that $\beta_0, \beta_t \in I$ and $\beta_0 \neq \beta_t$. By Lemma 2.3, $(r_0 r_i)^k = 1$ for some $i$ where $k = 3, 4, 6$. On the other hand, $i \in P \cup Q$ implies $(\alpha_0, \alpha_i) = 0$ or $(\alpha_i, \alpha_i) \leq (\alpha_0, \alpha_i)^2$ and hence $(r_0 r_i)^2 = 1$ or $r_0 r_i$ has an infinite order. We come to a contradiction.

**Corollary 2.5.** *Let $I$ be the subset of $\mathring{W}$ which is constructed in the Proof of Theorem 2.4. Then there exists a one-to-one correspondence between $\mathring{W}\alpha_0$ and $I$.*

*Proof.* Let $\widetilde{\mathring{W}}$ be as above. For each $\omega \in \mathring{W}$, there exists exactly one element $\omega' \in I$ such that $\omega\widetilde{\mathring{W}} = \omega'\widetilde{\mathring{W}}$. Define a map $\phi : \mathring{W}\alpha_0 \to I$ by $\phi(\omega\alpha_0) = \omega'$. Clearly $\phi$ is onto. We only need to prove that $\phi$ is one-to-one. For $\omega_1, \omega_2 \in \mathring{W}$, suppose $\phi(\omega_1\alpha_0) = \phi(\omega_2\alpha_0)$.

Then $\omega_1\widetilde{\mathring{W}} = \omega_2\widetilde{\mathring{W}}$. Thus $\omega_2^{-1}\omega_1 \in \widetilde{\mathring{W}}$, and hence $\omega_1\alpha_0 = \omega_2\alpha_0$.

**Theorem 2.6.** *Let $P$ and $Q$ be the same sets as in Theorem 2.4. If $P \cup Q = \{0, 1, \ldots, n\}$, then $T$ is a free Coxter group which is normal in $W$.*

*Proof.* Clearly $T$ is a normal subgroup of $W$. We need to show that $T$ is a free Coxter group. Let's enumerate all elements of $I$ by $I = \{\beta_1, \ldots, \beta_m\}$. We claim that $(\beta_i\alpha_0, \beta_j\alpha_0) \leq -1$ for $i \neq j$. In fact,

$$\begin{aligned}
(\beta_i\alpha_0, \beta_j\alpha_0) &= (\beta_j^{-1}\beta_i\alpha_0, \alpha_0) \\
&= (\alpha_0 + k_1\alpha_1 + \cdots + k_n\alpha_n, \alpha_0) \quad \text{where } k_i \in \mathbb{Z}_{\geq 0} \\
&= (\alpha_0, \alpha_0) + k_1(\alpha_1, \alpha_0) + \cdots + k_n(\alpha_n, \alpha_0) \\
&= 1 + \frac{k_1}{2}a_{01} + \cdots + \frac{k_n}{2}a_{0n} \\
&\leq 1 - \frac{1}{2}a_{i0}a_{0i} \quad \text{for some } i \in P \quad [3] \\
&= 1 - \frac{2(\alpha_0, \alpha_i)^2}{(\alpha_i, \alpha_i)} \\
&\leq 1 - 2 = -1.
\end{aligned}$$

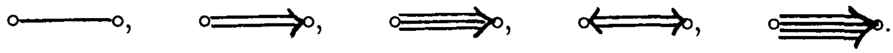
Since $(\beta_i\alpha_0, \beta_i\alpha_0) = (\beta_j\alpha_0, \beta_j\alpha_0) = (\alpha_0, \alpha_0)$, $(\beta_i\alpha_0, \beta_i\alpha_0)(\beta_j\alpha_0, \beta_j\alpha_0) = (\alpha_0, \alpha_0)^2 = 1 \leq (\beta_i\alpha_0, \beta_j\alpha_0)^2$. Hence $\langle r_{\beta_i\alpha_0} \mid \beta_i \in I \rangle$ is a free Coxter group by Lemma 2.2.

## 3. Hyperbolic case

**Lemma 3.1.** [3]. *Let* $A = (a_{ij})_{i,j=0}^{n}$ *be a generalized Cartan matrix of hyperbolic type.*

*Suppose* $a_{ij} \neq 0$ *and let* $S_2$ *be the subdiagram of* $S(A)$ *consisting of vertices* $i$ *and* $j$. *Then the following properties are satisfied:*

(a) *If* $n = 2$, *then* $S_2$ *is one of the following diagrams:*



(b) *If* $n = 3$, *then* $S_2$ *is one of the first three diagrams above.*

(c) *If* $n \geq 4$, *then* $S_2$ *is one of the first two diagrams above.*

(d) *If* $A$ *is of strictly hyperbolic type, then* $n \leq 4$.

**Corollary 3.2.** *Let* $A = (a_{ij})_{i,j=0}^{n}$ *be a generalized Cartan matrix of hyperbolic type and* $P$, $Q$ *be the same sets as in Theorem 2.4. If* $P \cup Q = \{0, 1, \dots, n\}$, *then* rank $A \leq 3$ *and* $T$ *is also a free Coxter group.*

*Proof.* It follows immediately from Theorem 2.4 and Lemma 2.2.

**Theorem 3.3.** *Let* $A$, $P$, $Q$ *be as in Corollary 3.2. If* $P \cup Q = \{0, 1, \dots, n\}$ *and* $S(A)$ *has no cycle, then the following properties are satisfied:*

(a) *If* $A$ *is of strictly hyperbolic type of* rank 3, *then* $|I| = |\mathring{W}|/2$.

(b) *If* $A$ *is of hyperbolic type of* rank 3 *which is not strictly hyperbolic, then* $I$ *is infinite.*

(c) *If* $A$ *is of hyperbolic type of* rank 2, *then* $T$ *is generated by the set* $\{r_0, r_1 r_0 r_1\}$.

*Proof.* Suppose that rank $A = 3$. Then we may assume that $P = \{0, 1\}$, $Q = \{2\}$. Then $\widetilde{\mathring{W}} = \{1, r_2\}$. Suppose $A$ is of strictly hyperbolic type. Then $\mathring{W}$is finite, and hence

$$|I| = |\mathring{W}/\widetilde{\mathring{W}}| = |\mathring{W}|/|\widetilde{\mathring{W}}| = |\mathring{W}|/2.$$

Suppose that $A$ is of affine type. Then $\mathring{W} = \{r_1(r_1 r_2)^m, (r_1 r_2)^m \mid m \in \mathbb{Z}\}$. Thus $\mathring{W}$is infinite and $|\widetilde{\mathring{W}}| = 2$. Therefore $I$ is infinite. Suppose rank $A = 2$. Then $\mathring{W} = \{1, r_1\}$ and hence $T = \langle r_0, r_1 r_0 r_1 \rangle$.

**Example.** *Let* $A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. *Then* $\mathring{W} = \{1, r_1, r_1 r_2, r_1 r_2 r_1, r_2, r_2 r_1\}$, $\widetilde{\mathring{W}} = \langle r_2 \rangle$ *and* $I = \{1, r_1, r_2 r_1\}$. $\mathring{W}\alpha_0 = \{\alpha_0, r_1\alpha_0, r_2 r_1 \alpha_0\}$. $T = \langle r_0, r_{r_1\alpha_0}, r_{r_2 r_1 \alpha_0} \rangle$.

**Theorem 3.4.** *Let $A$, $P$, $Q$ be as in Theorem 3.3. If $P \cup Q = \{0, 1, \ldots, n\}$ and $S(A)$ has a cycle, then there exists a one-to-one correspondence between $\mathring{W}\alpha_0$ and $\mathring{W}$.*

*Proof.* Since $S(A)$ has a cycle, rank $A = 3$ and $P = \{0, 1, 2\}$, $Q = \phi$. Hence $\widetilde{\mathring{W}} = \{1\}$. If $\omega_1\alpha_0 = \omega_2\alpha_0$ for $\omega_1$, $\omega_2 \in \mathring{W}$, then $\omega_2^{-1}\omega_1\alpha_0 = \alpha_0$ and hence $\omega_2^{-1}\omega_1 \in \widetilde{\mathring{W}} = \{1\}$. Therefore $\omega_1 = \omega_2$.

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