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HARMONIC GAUSS MAP AND HOPF FIBRATIONS

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ABSTRACT. A Gauss map of m-dimensional distribution on a Riemannian manifold M is called a harmonic Gauss map if it is a harmonic map from the manifold into its Grassmann bundle $G_m(TM)$ of m-dimensional tangent subspace. We calculate the tension field of the Gauss map of m-dimensional distribution and especially show that the Hopf fibrations on S^{4n+3} are the harmonic Gauss map of 3-dimensional distribution.

1. Introduction

To find an optimal vector fields and distributions on the round spheres is very interesting subject. Since a vector field on a Riemannian manifold M is a map from M to its tangent bundle TM as a graph, we can think about the best vector fields on M in two ways; the volume[5] and energy[10].

In [5] they showed that the Hopf vector field on S^3 as a graph in the unit tangent bundle with the Sasaki metric have the minimal volume in its homology class and no others. On higher dimensional spheres the Hopf vector fields are critical points of the volume functional but not even local minimum. By the another way, the Hopf vector fields on S^{2n+1} are harmonic maps from the sphere into the unit tangent bundle US^{2n+1}, i.e. the critical points of the energy functional [6]. However these are not energy minimizer since harmonic maps from spheres to compact manifolds are unstable.

Consider the another Hopf fibrations. These are not a vector fields, but 3-dimensional or 7-dimensional distribution on the round spheres. A m-plane distribution on a Riemannian manifold M assigns to each point $p \in M$ a m-dimensional subspace of the tangent space at p. Another way to view a m-plane distribution is a

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section of the Grassmann bundle $G_m(TM)$, or as the immersion $\xi : M \to G_m(TM)$, given by the graph of the section. We call this the Gauss map of m-dimensional distribution. Hence the Hopf fibrations on S^{4n+3} are the Gauss map of 3-dimensional distribution from S^{4n+3} to $G_3(TS^{4n+3})$. In this paper we will show that the Hopf fibrations on S^{4n+3} is harmonic as a graph in its Grassmann bundle of 3-dimensional tangent subspace. This work is the generalization of [6].

On the other hand, Ruh-Vilms[11] proved that the Gauss map of an isometric immersion $f : M^k \to R^n$ is harmonic if and only if f has parallel mean curvature vector. In his paper the Gauss map assigns to a point $p \in M$ the k dimensional subspace of R^n obtained from the parallel translation of f_*T_pM to the origin. It thus takes values in Grassmann bundle $G_k(n)$, endowed with an $O(n)$-invariant Riemannian metric. This work is generalized such that a submanifold of an Einstein space has a harmonic Gauss map if and only if its mean curvature field is parallel [12].

In this paper we want to prove the following theorem.

Main Theorem. *The Hopf fibrations on S^{4n+3} are the harmonic Gauss maps.*

First we need a harmonic equation of Gauss map of m-dimensional distribution to find the best distribution. The notion of harmonicity of these Gauss maps requires some Riemannian metric on the fibre bundles $G_m(TM)$. In section 2 we will see the geometry of Grassmann bundle[9]. Section 3 is devoted to set up the tension field of Gauss map of m-dimensional distribution. In section 4 we will prove the main theorem.

2. The geometry of Grassmannian bundle

Let M^n be a Riemannian manifold with m-dimensional distribution $\mathcal{F}$. Denote its $O(n)$-bundle of orthonormal frames by

$$O(n) \to M$$

on which live the canonical form and Levi-Civita connection, respectively,

$$\theta = (\theta^A), \quad \omega = (\omega^A_B), \quad \omega^A_B = -\omega^B_A.$$

Throughout this paper we use the following index conventions

$$1 \leq i,j,k \leq m, \quad m+1 \leq \alpha,\beta,\gamma \leq n, \quad 1 \leq A,B,C,D \leq n$$

where m is the dimension of the distribution, $1 \leq m \leq n$. The structure equations are

$$d\theta^A = -\omega_B^A \wedge \theta^B, \quad d\omega_B^A = -\omega_C^A \wedge \omega_B^C + \Omega_B^A, \tag{1}$$

and the curvature forms Ω_B^A is given by

$$\Omega_B^A = \frac{1}{2} R_{BCD}^A \theta^C \wedge \theta^D$$

where the R_{BCD}^A satisfy the usual symmetry relations of the Riemann curvature tensor.

Let

$$\pi : G_m(TM) \to M$$

denote the Grassmann bundle over M of m-dimensional tangent subspaces of TM. It is a fibre bundle over M associated to $O(M)$ with standard fibre the Grassmann manifold

$$G_m(n) = O(n)/O(m) \times O(n-m)$$

on which $O(n)$ acts on the left by multiplication. Since there exists a local cross section of the principal fibre bundle

$$O(m) \times O(n-m) \to O(n) \to G_m(TM),$$

we can pull the canonical forms θ^A and connection form ω^A down to get local forms on $G_m(TM)$. Then we can use this method to give a Riemannian metric on $G_m(TM)$.

Proposition 1[9,13]. *The quadratic tensor*

$$ds^2 = \sum_i \theta_i^2 + \sum_{i,\alpha} \omega_{i\alpha}^2$$

is globally defined metric tensor on $G_m(TM)$

Let $u_1, \cdots, u_n$ denote the standard basis of R^n and for the origin of $G_m(n)$ we choose the subspace of R^n spanned by $u_1, \cdots, u_m$, which denote

$$o = u_1 \wedge \cdots \wedge u_m.$$

If $U \subset G_m(TM)$ is an open subset containing o and

$$u : U \to O(M)$$

is any local section, then

$$\{\varphi^{\mathcal{A}} | \varphi^A = u^* \theta^A \text{ if } \mathcal{A} = A, \quad \varphi^\mu = u^* \omega_i^\alpha \text{ if } \mathcal{A} = \mu = (\alpha i)\}$$

is an orthonormal coframe from ds^2 on U. From the structure equation of $O(M)$, we find that the pull-back by u^* of the forms

$$\begin{aligned}
\varphi_B^A &= \omega_B^A + \frac{1}{2} R_{iAB}^\alpha \omega_\alpha^i, \\
\varphi_B^\mu &= \frac{1}{2} R_{iBA}^\alpha \theta^A = -\varphi_\mu^B, \quad \mu = (\alpha i), \\
\varphi_\nu^\mu &= \delta_{\alpha\beta} \omega_j^i + \delta_{ij} \omega_\beta^\alpha, \quad \mu = (\alpha i), \nu = (\beta j)
\end{aligned} \tag{2}$$

gives the Levi-Civita connection forms of ds^2 with respect to this orthonormal coframe field.

3. Tension field of Gauss map

Let $\psi : M \to G_m(M)$ be a Gauss map of m-dimensional distribution. Define $F_B^{\mathcal{A}}$ by

$$\psi^*(\varphi^{\mathcal{A}}) = \sum_{B=1}^n F_B^{\mathcal{A}} \theta^B. \tag{3}$$

If we take the exterior differentiation of (3) and use the structure equations (1) we get;

$$\sum_B (dF_B^{\mathcal{A}} + F_B^{\mathcal{B}} \varphi_{\mathcal{B}}^{\mathcal{A}} - F_C^{\mathcal{A}} \omega_B^C) \wedge \theta_B = 0.$$

Thus, by the Cartan lemma we have the fundamental tensor $F_{BC}^{\mathcal{A}}$ of mapping ψ such that

$$\sum_C F_{BC}^{\mathcal{A}} \theta^C = dF_B^{\mathcal{A}} + F_B^{\mathcal{B}} \varphi_{\mathcal{B}}^{\mathcal{A}} - F_C^{\mathcal{A}} \omega_B^C. \tag{4}$$

We define $II_f = \sum_{A,B,C} F^A_{BC}\theta^B \otimes \theta^C \otimes e_A$ to be the second fundamental form of the mapping, and the trace of $\tau(\psi) = \sum_B F^A_{BB} \otimes e_A$ to be the tension of the map ψ. When $\tau(\psi) = 0$, ψ is called a harmonic map [1,2][8].

By using the equation [2] and [4], we can calculate the tangential part of tension field $\tau^H = \sum_A F^A_{BB}e_A$ and the vertical part of tension field $\tau^V = \sum_\mu F^\mu_{BB}e_\mu, \mu = (\alpha i)$. Using the Christoffel symbols we can write $\omega^A_B = \Gamma^A_{BC}\theta^C$.

Proposition 2. *Let ψ be a Gauss map of m-dimensional distribution from M to $G_m(TM)$. ψ is harmonic if and only if*

$$\tau^H = \sum_A \sum_{i,B,\alpha} \Gamma^i_{\alpha B} R^\alpha_{iAB} e_A = 0,$$
$$\tau^V = \sum_\mu \Gamma^\alpha_{iBB}, \mu = (\alpha i),$$

where $\sum_C \Gamma^\alpha_{iBC}\theta^C = d\Gamma^\alpha_{iB} + \Gamma^\alpha_{jB}\omega^i_j + \Gamma^\beta_{iB}\omega^\alpha_\beta - \Gamma^\alpha_{iD}\omega^D_B$.

Proof. We can find that

$$F^A_B = \delta^A_B, \quad F^\mu_A = \Gamma^\alpha_{iA}, \mu = (\alpha i).$$

Hence by [2] and [4] the covariant derivative of the tensor F^A_B is

$$\begin{aligned}\sum_C F^A_{BC}\theta^C &= dF^A_B + F^D_B\varphi^A_D + F^\mu_B\varphi^A_\mu - F^A_D\omega^D_B \\ &= \omega^A_B + \frac{1}{2}R^\alpha_{iAB}\omega^i_\alpha - \frac{1}{2}\Gamma^\alpha_{iB}R^\alpha_{iAD}\theta^D - \omega^A_B.\end{aligned}$$

Thus

$$F^A_{BC} = \frac{1}{2}\sum_{\alpha i}(R^\alpha_{iAB}\Gamma^i_{\alpha C} - R^\alpha_{iAC}\Gamma^\alpha_{iB},$$

and so

$$\sum_B F^A_{BB} = R^\alpha_{iAB}\Gamma^i_{\alpha B}.$$

By the same calculation,

$$\begin{aligned}\sum_C F^\mu_{BC}\theta^C &= dF^\mu_B + F^D_B\varphi^\mu_D + F^\nu_B\varphi^\mu_\nu - F^\mu_D\omega^D_B \\ &= d\Gamma^\alpha_{iA} + \frac{1}{2}\delta^D_B R^\alpha_{iDC}\theta^C + \Gamma^\beta_{jB}(\delta_{\alpha\beta}\omega^i_j + \delta_{ij}\omega^\alpha_\beta) - \Gamma^\alpha_{iD}\omega^D_B \\ &= d\Gamma^\alpha_{iB} + \frac{1}{2}R^\alpha_{iBC}\theta^C + \Gamma^\alpha_{jB}\omega^i_j + \Gamma^\beta_{iB}\omega^\alpha_\beta - \Gamma^\alpha_{iD}\omega^D_B.\end{aligned}$$

Since

$$\sum_C \Gamma^\alpha_{iBC}\theta^C = d\Gamma^\alpha_{iB} - \Gamma^\alpha_{iD}\omega^D_B + \Gamma^\alpha_{jB}\omega^i_j + \Gamma^\beta_{iB}\omega^\alpha_\beta,$$

$$F^\mu_{BB} = \sum_m u\Gamma^\alpha_{iBB}, \mu = (\alpha i).$$

The τ^H and τ^V have the following meaning. First,

$$\begin{aligned}\tau^H &= \sum_{B,A} F^A_{BB} e_A,\\ &= -\sum_{B,A,i,\alpha} \Gamma^i_{\alpha B} R^\alpha_{iAB} e_A,\\ &= -\sum_{B,A,i,\alpha} (\Gamma^i_{\alpha B} R(e_\alpha, e_i)e_B, e_A)e_A,\\ &= -\sum_{B,i} (R(\nabla_{e_B} e_i, e_i)e_B),\\ &= -\sum_i \text{ trace } R(\nabla_* e_i, e_i)*,\end{aligned}$$

where $\nabla_{e_B} e_A = \Gamma^A_{CB} e_C$. Second, we will use the notation of [12]. Let $\rho^\perp$ be the orthogonal projection from the section of TM to $\mathcal{F}^\perp$ and $\bar\nabla$ denotes the $\mathcal{F}^\perp$-component of the Levi-Civita connection ∇ of (M, g) and $\bar\nabla\rho^\perp$ is defined by

$$\bar\nabla_X \rho^\perp(v) = \bar\nabla_X v^\perp - \rho^\perp(\nabla_X v).$$

If $\bar\nabla^2$ is rough Laplacian such that $\bar\nabla^2 = \sum_B \bar\nabla_{e_B}\bar\nabla_{e_B} - \bar\nabla_{\bar\nabla_{e_B} e_B}$,

$$\begin{aligned}\sum_i (\bar\nabla^2\rho^\perp)(e_i) &= \sum_i\sum_B (\bar\nabla_{e_B}\bar\nabla_{e_B}\rho^\perp - \nabla_{\nabla_{e_B} e_B}\rho^\perp)(e_i),\\ &= \sum_{i,B} [\bar\nabla_{e_B}((\bar\nabla_{e_B}\rho^\perp)(e_i)) - (\bar\nabla_{e_B}\rho^\perp(\nabla_{e_B} e_i))]\\ &\quad - [\bar\nabla_{\bar\nabla_{e_B} e_B}(\rho^\perp(e_i)) - \rho^\perp(\nabla_{\nabla_{e_B} e_B} e_i)],\\ &= \sum_{i,B} [-2\bar\nabla_{e_B}(\rho^\perp(\nabla_{e_B} e_i)) + \rho^\perp(\nabla_{e_B}\nabla_{e_B} e_i) + \rho^\perp(\nabla_{\nabla_{e_B} e_B} e_i)],\\ &= \tau^V.\end{aligned}$$

Theorem 3. *ψ is a harmonic map if and only if*

$$\sum_i (\text{ trace } R(\nabla_* e^i, e^i)*)^H = 0, \quad \bar{\nabla}^2 \rho^\perp|_{\mathcal{F}} = 0.$$

Hence this result is a generalization of the tension field equation of the unit vector fields[6].

4. Harmonicity of Hopf Fibrations in S^{4n+3}

Let C be the Complex numbers, H the Quaternions and Ca the Cayley numbers. The Hopf fibrations are

$$\begin{aligned} S^1 \to & S^{2n+1} \to CP^n = \text{ Complex projective } n \text{ space,} \\ S^3 \to & S^{4n+3} \to HP^n = \text{ Quaternionic projective } n \text{ space,} \\ S^7 \to & S^{15} \to S^8. \end{aligned}$$

This fibrations have many beautiful properties. For example, their fibres are parallel in the sense of having constant distance from one another [4]. This actually characterizes the Hopf fibrations among all fibrations of round spheres by great subspheres, as was proved by [3] and other many papers.

In the 3-dimensional fibrations in S^{4n+3}, this admit a Sasakian 3-stucture $\{e_1, e_2, e_3\}$ [7]. Hence this three structure is mutually orthogonal and satisfies the conditions

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = 2e_1, \quad [e_3, e_1] = 2e_2.$$

This means that

$$\begin{aligned} \nabla_{e_i} e_i &= 0 \\ \nabla_{e_2} e_1 &= e_3 = -\nabla_{e_1} e_2, \\ \nabla_{e_3} e_2 &= e_1 = -\nabla_{e_2} e_3, \\ \nabla_{e_1} e_3 &= e_2 = -\nabla_{e_3} e_1. \end{aligned}$$

Hence as in the case of [6], we can choose a standard basis

$$\{\hat{e}_i, \hat{e}_{4n+1} = e_1, \hat{e}_{4n+2} = e_2, \hat{e}_{4n+3} = e_3\}, i = 1, \cdots 4n$$

such that e_1, e_2, e_3 are Sasakian 3-structure and $\{\hat{e}_1, \cdots \hat{e}_{4n}\}$ is the horizontal lift of the normal unitary basis $\{f_i\}$ of HP^n, i.e. $\nabla_{f_i} f_j = 0$ and

$$If_{4i} = f_{4i+1}, \quad Jf_{4i} = f_{4i+2}, \quad Kf_{4i} = f_{4i+3}$$

where I, J, K are the Quaternionic structure. Also

$$\begin{array}{llll}
\nabla_{\hat{e}_{4i+1}} e_1 = \hat{e}_{4i+2}, & \nabla_{\hat{e}_{4i+2}} e_1 = -\hat{e}_{4i+1}, & \nabla_{\hat{e}_{4i+3}} e_1 = \hat{e}_{4i+4}, & \nabla_{\hat{e}_{4i+4}} e_1 = -\hat{e}_{4i+3}, \\
\nabla_{\hat{e}_{4i+1}} e_2 = \hat{e}_{4i+3}, & \nabla_{\hat{e}_{4i+2}} e_2 = -\hat{e}_{4i+4}, & \nabla_{\hat{e}_{4i+3}} e_2 = -\hat{e}_{4i+1}, & \nabla_{\hat{e}_{4i+4}} e_2 = \hat{e}_{4i+2}, \\
\nabla_{\hat{e}_{4i+1}} e_3 = \hat{e}_{4i+4}, & \nabla_{\hat{e}_{4i+2}} e_3 = \hat{e}_{4i+3}, & \nabla_{\hat{e}_{4i+3}} e_3 = -\hat{e}_{4i+2}, & \nabla_{\hat{e}_{4i+4}} e_3 = -\hat{e}_{4i+1}.
\end{array}$$

Using these formulae of covariant derivatives, we can calculate the tension field of ψ in S^{4n+3} and show that its horizontal and vertical components both vanish. The horizontal component is

$$\begin{aligned}
\tau^H &= \sum_{i=1}^{3} \sum_{k=1}^{4n+3} (R(\nabla_{\hat{e}_k} e_i, e_i)\hat{e}_k) \\
&= \sum_{i,k} (e_i, \hat{e}_k)\nabla_{\hat{e}_k} e_i - (\hat{e}_k, \nabla_{\hat{e}_k} e_i) e_i \\
&= \sum_{i} \nabla_{e_i} e_i = 0.
\end{aligned}$$

We now compute the vertical component.

$$\begin{aligned}
\tau^V &= \sum_{i=1}^{3} (\bar{\nabla}^2 \rho^\perp)(e_i) \\
&= \sum_{i} \sum_{B=1}^{4n+3} (\bar{\nabla}_{\hat{e}_B} \bar{\nabla}_{\hat{e}_B} \rho^\perp - \nabla_{\nabla_{\hat{e}_B} \hat{e}_B} \rho^\perp)(e_i) \\
&= \sum_{i,B} [-2\bar{\nabla}_{\hat{e}_B} (\rho^\perp(\nabla_{\hat{e}_B} e_i)) + \rho^\perp(\nabla_{\hat{e}_B} \nabla_{\hat{e}_B} e_i) + \rho^\perp(\nabla_{\nabla_{\hat{e}_B} \hat{e}_B} e_i)] \\
&= 0
\end{aligned}$$

Therefore we conclude the following theorem.

Theorem 4. *The Hopf fibrations on S^{4n+3} are the harmonic Gauss maps.*

In [6] they proved that on S^3 a smooth unit vector field ξ is a harmonic map into the unit tangent bundle US^3 with the Sasaki metric if and only if it is the tangent vector field of the Hopf fibration. In S^7 we do not know about that.

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