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A VERTEX PROPERTY OF REAL FUNCTION ALGEBRAS

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ABSTRACT. We investigate a chain of properties of real function algebras along the analogous proofs of the complex cases such as the fact that any real function algebra which is both maximal and essential is pervasive. And some properties of real function algebras with a vertex property will be discussed.

1. Introduction

Comparing with complex Banach algebras, the study of *real* Banach algebras started quite late and not many researches have been done because the structure of real algebras is much more complicated to handle with than that of complex algebras. The first mathematician who studied real Banach algebras systematically is known to be L. Ingelstam [I2] in the early 1960's. A typical example of complex (real, respectively) Banach algebras is $C(X)(C_{\mathbf{R}}(X)$, respectively), the algebra of complex-valued (real-valued, respectively) continuous functions on a compact Hausdorff space X.

Usually, a real function algebra can be understood as a real subalgebra of $C_{\mathbf{R}}(X)$. In this note we will consider a function algebra which lies in the complex algebra $C(X)$ but has a real linear space structure under certain restriction. More precisely, by a *complex function algebra* B on X we mean a uniformly closed complex subalgebra of $C(X)$ which separates the points of X and contains the constant functions. Here point separation means that if x_1 and x_2 are distinct points of X then $f(x_1) \neq f(x_2)$ for some function f in B. On the other hand, by a *real function algebra* A on (X, τ) we mean a uniformly closed real subalgebra of the real Banach algebra $C(X, \tau)$ which separates the points of X and contains the constant

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functions. Here τ is an involution on X, that is, a homeomorphism on X such that $\tau \circ \tau(x) = x$ for all $x \in X$, and $C(X, \tau) = \{f \in C(X) : f \circ \tau = \overline{f}\}$, where $\overline{f}$ means the complex conjugate of f. Note that if τ is the identity map then we have $C(X, \tau) = C_{\mathbf{R}}(X)$.

It is not difficult to see that $C(X) = C(X, \tau) + iC(X, \tau)$ and that $C(X, \tau) = C_R(X, \tau) + iC_S(X, \tau)$, where $C_R(X, \tau) = \{u \in C_{\mathbf{R}}(X) : u \circ \tau = u\}$ and $C_S(X, \tau) = \{v \in C_{\mathbf{R}}(X) : v \circ \tau = -v\}$. The algebra $C(X, \tau)$ plays the role of $C(X)$ in this real function algebra theory.

Properties of real function algebras have been studied by several authors since early 1980's and most updated results can be found in the book written by S. Kulkarni and B. Limaye [K-L]. In Section 2, some properties of real function algebras will be developed along the lines of complex cases, and we will show a structural difference between real and complex cases. In Section 3, a vertex property of real Banach algebras will be studied and we will show that this property lies between antisymmetry and essentiality under some restrictions.

Materials in this article can be found mostly in the author's thesis [H], but we give here detailed descriptions to make it self-contained.

2. Properties of Real Function Algebras

The following chain of definitions and properties for real function algebras on (X, τ) are adopted from the corresponding ones for complex function algebras on X. Proofs of most of the results in this section can be found in [H] and follow closely the analogous proofs in the complex cases given in [H-S] and [L].

We say that a subset Y of X is *τ-symmetric* if $\tau(Y) = Y$, and we denote by $\|\ \|_F$ the supremum norm on a closed subset F.

The corresponding ones for complex cases of the following definitions and related facts can be found in [H-S] and [L].

Definition 2.1. A real function algebra A on (X, τ) is said to be

(1) *pervasive* if the restriction algebra $A|_F$ is dense in $C(F, \tau_F)$ for every τ-symmetric proper compact subset F of X. Here, τ_F is the restriction of τ to F;

(2) *analytic* if the only function in A which vanishes identically on a non-empty τ-symmetric open set U in X is 0;
(3) *essential* if the only closed ideal of $C(X,\tau)$ in A is the trivial ideal (0);
(4) *maximal* if for any real function algebra B on (X,τ) with $A \subset B \subset C(X,\tau)$ we have either $A = B$ or $B = C(X,\tau)$.

Lemma 2.2. *Let A be a real function algebra on (X,τ) and let $f \in C(X,\tau)$. Then the closed ideal I generated by f in $C(X,\tau)$ is $I = \{g \in C(X,\tau) : g|_F = 0\}$, where $F = f^{-1}(0)$.*

Proof. It is easy to show that $I \subset \{g \in C(X,\tau) : g|_F = 0\}$. On the other hand, let $g \in C(X,\tau)$ with $g|_F = 0$. Since $F \subset g^{-1}(0)$, given $\epsilon > 0$ we can take a τ-symmetric open neighborhood U of F such that $|f|, |g| < \epsilon$ on U. Take a τ-symmetric open set V such that $F \subset V \subset \overline{V} \subset U$, where $\overline{V}$ is the closure of V. Choose $h \in C(X,\tau)$ such that $h = 0$ on V, $h = g$ on $X \setminus U$, and $||h - g||_X < \epsilon$.

Now choose a τ-symmetric open set W such that $F \subset W \subset \overline{W} \subset V \subset \overline{V} \subset U$. Define $k \in C(X,\tau)$ by $k = hf^{-1}$ on $X \setminus W$ and $k = 0$ on V. Then $h = fk \in I$. Since $||h - g||_X < \epsilon$ and I is closed, $g \in I$. □

Theorem 2.3. *If A is a proper pervasive real function algebra on (X,τ), then A is analytic.*

Proof. Let f be a function in A which vanishes on a non-empty τ-symmetric open set U of X. We will show $f = 0$ on X.

Let $K = X \setminus U$. Then K is a proper τ-symmetric compact subset of X. Since A is pervasive, for any $g \in C(X,\tau)$ there exists a sequence $\{f_n\}$ in A which converges uniformly to g on K. Then $ff_n \to fg$ uniformly on X since $f = 0$ on $X \setminus K = U$. Thus $fg \in A$ for any $g \in C(X,\tau)$, and hence A contains the closed ideal I generated by f in $C(X,\tau)$.

Now, put $F = f^{-1}(0)$. Then F is τ-symmetric, and $I = \{h \in C(X,\tau) : h|_F = 0\} \subset A$ by the above lemma.

We first claim that $A|_F$ is closed in $C(X,\tau)$. To show this, let $g \in C(X,\tau)$ and let $\{g_n\}$ be a sequence in A such that $g_n \to g$ uniformly on F. Choose a sequence $\{h_n\}$ in $C(X,\tau)$ such that $h_n = g_n$ on F and h_n converges uniformly to a function $h \in C(X,\tau)$. Then $h_n - g_n = 0$ on F and hence $h_n - g_n \in I \subset A$. Since $g_n \in A$, we have $h_n \in A$ and therefore $h \in A$. But $h = g$ on F, and so $h - g \in I \subset A$. Thus $g \in A$. This proves the claim.

Now suppose $F \neq X$. Then $A|_F$ is dense in $C(F, \tau_F)$, and therefore $A|_F = C(F, \tau_F)$ by the claim. Let $g \in C(X, \tau)$. Then $g|_F \in C(F, \tau_F) = A|_F$, so $g|_F = h|_F$ for some $h \in A$. Since $g - h = 0$ on F, $g - h \in I \subset A$, so $g \in A$. Thus $A = C(X, \tau)$, which contradicts the fact that A is a proper subalgebra of $C(X, \tau)$.

Therefore, $F = X$ and $f = 0$ on X. □

It was E. Bishop [B] who introduced and studied the concept of antisymmetry systematically to generalize the Stone-Weierstrass Theorem. The following real version of the definition can be seen in [H] and [K-L].

Definition 2.4. A real subalgebra A of $C(X)$ is said to be

(1) *partially antisymmetric* if A contains no non-constant real functions;
(2) *antisymmetric* if A contains no non-constant real functions and no non-constant purely imaginary functions.

Note that antisymmetry implies partial antisymmetry, but the converse is not true as shown in the following Example 2.6. Note also that for complex algebras they are the same concept.

Theorem 2.5. *Let A be a real function algebra on (X, τ).*

(1) *If A is analytic, then A is an integral domain.*
(2) *If A is an integral domain, then A is partially antisymmetric.*
(3) *If A is partially antisymmetric, then A is essential.*

Proof. (1) Let f and g be non-zero functions in A. Choose $x_0 \in X$ such that $f(x_0) \neq 0$, and then choose an open neighborhood U of x_0 such that $|f| > 0$ on U. Then for $x \in U$ we have $|f(\tau(x))| = |\overline{f(x)}| = |f(x)| > 0$, so $|f| > 0$ on $\tau(U)$. Let $U_1 = U \cup \tau(U)$. Then $x_0 \in U_1$, $\tau(U_1) = U_1$, and $|f| > 0$ on U_1.

Since A is analytic and since $g \neq 0$, there exists $x_1 \in U_1$ such that $g(x_1) \neq 0$. Then $(fg)(x_1) = f(x_1)g(x_1) \neq 0$, so $fg \neq 0$. Therefore, A is an integral domain.

(2) Put $B = \{f \in A | \overline{f} \in A\}$, the *symmetric* part of A. Then B is a uniformly closed real subalgebra of $C(X, \tau)$ which is conjugate-closed and contains the constant functions. So by the Stone-Weierstrass Theorem for real algebras, $B = C(Y, \mu)$ for some compact Hausdorff space Y and an involution μ on Y. Since B is an integral domain, Y must be a two-point set. (If τ is the identity map, then Y is a singleton.) Therefore, A is partially antisymmetric.

(3) Suppose that A is not essential. Then A contains a non trivial closed ideal of $C(X,\tau)$. Hence A has a non-constant real function, so A is not partially antisymmetric. □

It is known that a complex function algebra on X is antisymmetric if it is an integral domain. But, for real function algebras on (X,τ) an intgral domain may not be antisymmetric as in the following simple example.

Example 2.6. Let $X = \{0,1\}$ and define $\tau(0) = 1$ and $\tau(1) = 0$. Then $C(X,\tau)$ is an integral domain, which is partially antisymmetric but is not antisymmetric.

Theorem 2.7. *If a real function algebra A on (X,τ) is both maximal and essential, then A is pervasive.*

Proof. Let F be a non-empty τ-symmetric compact proper subset of X and let $B = \{g \in C(X,\tau) : g|_F \in A_F\}$, where A_F is the uniform closure of $A|_F$ in $C(F,\tau_F)$. Then B is a closed subalgerba of $C(X,\tau)$ containing A.

Choose $g \in C(X,\tau)$ such that $g|_F = 0$. Then $g \in B$, and the closed ideal I generated by g in $C(X,\tau)$ is contained in B by the same argument as in the proof of Theorem 2.3. But I is not contained in A since A is essential. Thus $A \neq B$, so by the maximality of A we have $B = C(X,\tau)$. Therefore, $A_F = B|_F = C(F,\tau_F)$, and hence $A|_F$ is dense in $C(F,\tau_F)$. □

3. Real Function Algebras with a Vertex Property

A *vertex* of a convex set K is a point x_0 in the boundary of K with the property that no straight line through x_0 is a tangent of K. Here, a complex straight line is a real two-dimensional affine subspace. The following definition is adopted from L. Ingelstam [I1], which was originated by H. Bohnenblust and S. Karlin [B-K]: Let E be a (real or complex) Banach space with a norm N. A point $u \in E$ is called a *vertex of the unit sphere* if $\mathrm{N}(u) = 1$ and $\psi(x) = 0$ only if $x = 0$. Here, $\psi(x) = \max\{\Phi(\theta x) : |\theta| = 1\}$, where $\Phi(y)$ is the Gauteau differential

$$\Phi(y) = \lim_{\alpha \to 0^+} \frac{\mathrm{N}(u + \alpha y) - \mathrm{N}(u)}{\alpha}$$

and θ is chosen to be a real or complex number according to the (real or complex) space E.

By a *natural norm* of a commutative normed algerba with the identity e, we mean a norm satisfying $||xy|| \le ||x|| ||y||$ and $||e|| = 1$.

Definition 3.1. A Banach algebra with identity is said to have a *vertex property* if for all natural norms defining the topology, the identity is a vertex of the unit sphere.

Lemma ([I1, Theorem 2]). *A real commutative Banach algebra E with identity has a vertex property if and only if* $\exp(\alpha x)$ *is an unbounded function of the real variable α for every $x \neq 0$ in E.*

Proposition 3.2. *Let A be a uniformly closed real subalgebra of $C(X)$ with the constant functions. Then the following are equivalent:*

(1) *A has a vertex property.*
(2) *A has no non-zero purely imaginary functions.*

Proof. (1) implies (2): Suppose (2) is not true. Then there exists $f \in A$ such that $f = iv$ with $0 \neq v \in C_{\mathbf{R}}(X)$. Since Re$f$, the real part of f, is 0, for any $\alpha \in \mathbf{R}$ we have $||\exp(\alpha f)||_X = \sup\{\exp(\alpha \text{Re} f(x)) : x \in X\} = 1$, so $\exp(\alpha f)$ is bounded for all $\alpha \in \mathbf{R}$. Then by the lemma, A fails to have a vertex property.

(2) implies (1): Suppose (2) is true. Let $0 \neq f \in A$ be given, and let $x_0 \in X$ be such that Re$f(x_0) \neq 0$. Then as $\alpha \to \infty$ or $\alpha \to -\infty$

$$||\exp(\alpha f)||_X \ge \exp(\alpha \text{Re} f(x_0)) \to \infty,$$

so $\exp(\alpha f)$ is an unbounded function of $\alpha \in \mathbf{R}$. Then by the lemma again A has a vertex property. □

Corollary 3.3. *Let A be a uniformly closed real subalgebra of $C(X, \tau)$ which contains the constant functions. Then the following are equivalent:*

(1) *A has a vertex property.*
(2) *A has no non-constant purely imaginary functions.*

Proof. (1) implies (2) is obvious, so we will show (2) implies (1): Let $f \in A$ be a purely imaginary function. Then $f = i\alpha$ for some $\alpha \in \mathbf{C}$ by (2). Since $i\alpha = f = \overline{f \circ \tau} = -i\alpha$, we have $\alpha = 0$. Hence we have (1) by the above proposition. □

This corollary shows that for real function algebras on (X, τ), the counter part of partial antisymmetry is a vertex property in the sense that both of these properties imply antisymmetry. We rephrase this by

Corollary 3.4. *If a partially antisymmetric real function algebra A on (X, τ) has a vertex property, then A is antisymmetric, and vice versa.*

Comparing with the implication of antisymmetry to essentiality *via* partial antisymmetry, we would suspect the implication *via* vertex property. The following theorem shows that this implication depends on the topological structure of the diagonal of τ.

Theorem 3.5. *Let A be a real function algebra on (X, τ). Suppose τ is not the identity map, and suppose that $F = \{x \in X : \tau(x) = x\}$ has an empty interior. If A has a vertex property, then A is essential.*

Proof. Suppose A is not essential. Then A contains a non-trivial closed ideal I of $C(X, \tau)$. We will show that I contains a non-zero purely imaginary function. Then by the above proposition, A fails to have a vertex property.

Choose any $f = u + iv \neq 0$ in I, where $u \in C_R(X, \tau)$ and $v \in C_S(X, \tau)$.

Case 1. $u = 0$: Then trivially $0 \neq f = iv \in A$ is purely imaginary.

Case 2. $u \neq 0, v \neq 0$: Put $g = iv\overline{f} \in C(X, \tau)$. Then $fg \in I \subset A$, $fg = iv|f|^2 \neq 0$, and fg is purely imaginary.

Case 3. $u \neq 0, v = 0$: Then $f = u \neq 0$. Choose $x_0 \in X$ such that $f(x_0) = u(x_0) \neq 0$.

(i) If $x_0 \notin F$, we can choose $v_1 \in C_S(X, \tau)$ such that $v_1(x_0) \neq 0$ since $C(X, \tau)$ separates the points of X. Then $iv_1 f = iuv_1 \neq 0$ and $iv_1 f \in I \subset A$ is purely imaginary.

(ii) If $x_0 \in F$, we can choose a neighborhood U of X_0 such that $|f| > 0$ on U. Since F is a τ-symmetric proper closed subset of X and since F has an empty interior, we have $U \setminus F \neq \emptyset$, and hence we can choose $x_1 \in U \setminus F$ such that $f(x_1) = u(x_1) \neq 0$. This reduces to the case (i).

This proves the theorem. □

In the above theorem, the assumption that $F = \{x \in X : \tau(x) = x\}$ has an empty interior is necessary as we see in the following example.

Example 3.6. Let $X = \overline{\Delta} \cup [1, 2]$ where $\Delta = \{z \in \mathbf{C} : |z| < 1\}$, and define $\tau(z) = \overline{z}$ for $z \in X$. Then $F = \{z \in X : \tau(z) = z\} = [-1, 2]$ has a non-empty interior. Put $A = \{f \in C(X, \tau) : f|_\Delta \text{ is analytic}\}$. Then A has a vertex property but is not essential since the ideal $I = \{f \in C(X, \tau) : f|_\Delta = 0\}$ of $C(X, \tau)$ is contained in A.

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