J. Korea Soc. Math. Ed. Series B: The Pure and Applied Mathematics
4, No. 1 (1997) pp. 19–26

FIXED POINT PROPERTY AND COMPLETENESS OF ORDERED SETS

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ABSTRACT. In this paper, we characterize the existence of fixed points of a multivalued function by the existence of complete preorder on the given domain. Also we investigate relations between the completeness of a given order and the fixed point property of some multivalued functions.

1. Introduction

Let X be a partially ordered set. It is well-known from Zorn's lemma that if X satisfies one of

(A) every nonempty chain in X has an upper bound,
(B) every nonempty chain in X has a least upper bound,
(C) every nonempty well-ordered subset of X has an upper bound, or
(D) every nonempty well-ordered subset of X has a least upper bound,

then X has a maximal element.

If (A)((B), resp.) holds, then X is said to be *inductive* (*complete*, resp.). Note that (B) and (D) are logically equivalent. It is known that existence theorems of maximal elements in some ordered sets can be reformulated to various types of fixed point theorems (see Park [7]). One of them is the following Zermelo's fixed point theorem;

Received by the editors March 13, 1997 and, in revised form, May 23, 1997.
1991 *Mahtematics Subject Classification.* 06A06, 06A23.
Key words and phrases. preorder, inductive, complete, order extremal, normal order structure, fixed point.
This paper was supported by the grants for Professors of Sungshin Women's University in 1996.

Theorem [3, Theorem I. 2. 5]. *Let $(X, \leq)$ be a complete partially ordered set and $f : X \to X$ be a self-function satisfying*

$$\text{for all } x \in X,\ x \leq f(x) \tag{S1}$$

then f has a fixed point, that is, there is an $x \in X$ such that $x = f(x)$.

There have been many efforts to characterize the completeness of ordered sets by fixed point properties of some self-functions. Tarski [11] and Davis [2] proved that the completeness of a lattice is equivalent to the existence of fixed points of increasing self-functions. And Tasković [12] proved that a partially ordered set is complete if and only if every self-function satisfying (S1) has a fixed point.

On the other hand, Smarzewski [8] characterized the fixed point property as follows ;

Theorem [8, Theorem 1]. *Let X be a nonempty set and $f : X \to X$ be a self-function. Then f has a fixed point if and only if there exists a preorder $\leq$ on X such that $(X, \leq)$ has normal order structure and f satisfies (S1).*

Note that $(x, \leq)$ has normal order structure if and only if X is complete, if $\leq$ is a partially ordered set.

Smithson [9, 10] obtained some fixed point theorems for multivalued functions satisfying some increasing conditions. One of them is as follows;

Theorem [9, Proposition 1. 6]. *Let $(X, \leq)$ be a partially ordered set satisfying (D) and $F : X \to 2^X \setminus \{\emptyset\}$ be a multivalued function. Suppose that*

$$\text{for all } x \in X,\ \text{there exits a } y \in F(x) \text{ such that } x \leq y \tag{M1}$$

then F has a fixed point, that is, there is an $x \in X$ such that $x \in F(x)$.

In this paper, we obtain a characterization of the fixed point property for multivalued functions. We also characterize the completeness of ordered sets by fixed point property of multivalued functions. And some fixed point theorems for multivalued and single valued functions are proved.

2. A Characterization of Fixed Point Property

Let X be a nonempty set. A reflexive and transitive relation $\leq$ on X is called a *preorder*. Further, if $\leq$ is antisymmetric, then $\leq$ is called a *partial order*. A pair $(X, \leq)$ of a set X with a preorder (partial order, resp.) $\leq$ is called a *preordered set* (*partially ordered set,* resp.).

Let $(X, \leq)$ is a preordered set. The terms *chain, well-ordered subset, upper bound* can be defined as usual. For $x \in X$, we denote $S(x) = \{y \in X \mid x \leq y\}$. For $x \in X$and $A \subset X$, we call x a *maximal element* of A if

$$x \in A \text{ and if } y \in A,\ x \leq y \text{ then } y \leq x.$$

And x is called a *least upper bound* of A if

$$x \text{ is an upper bound of } A, \text{and if } y \text{ is an upper bound of } A,\ x \leq y.$$

Max(A) (Sup(A), resp.) will denote the set of all maximal elements (least upper bounds, resp.) of A. If Sup(A) $= \{x_0\}$ is a singleton, then we denote $x_0 = \sup A$.

X is said to be *inductive*(*complete,* resp.) if every nonempty chain in X has an upper bound (least upper bound, resp.).

Let $(X, \leq)$ be a preordered set. If we define a relation $\sim$ on X by

$$x \sim y \Longleftrightarrow x \leq y \text{ and } y \leq x,$$

then $\sim$ is an equivalence relation and $Y = X/\sim$ is a partially ordered set. For $x \in X$, $[x]$ will denote the equivalence class of x. It is easy to show that $[x]$ is a maximal element of Y if and only if x is a maximal element of X. Furthermore, if X is inductive(complete, resp.) then so is Y.

A nonempty subset E of X is said to be *order extremal* if

(i) for all $x, y \in E$, $x \sim y$, and
(ii) if $x \in E$, $y \in X$ and $x \leq y$, then $y \in E$.

If X is complete and every order extremal subset of X is a singleton, then we say that X has *normal order structure* [8]. Obviously, if X is partially ordered, then X has normal order structure if and only if X is complete.

Let $(X, \leq)$ be a preordered set. For a single valued function $f : X \to X$, we consider the following conditions;

(S1) for all $x \in X$, $x \leq f(x)$.

(S2) there is an $e \in X$ such that $e \leq f(e)$.

(S3) if $x_1 \leq x_2$, then $f(x_1) \leq f(x_2)$.

(S4) if C is a well-ordered subset of X such that $f(x) = x + 1$ for all $x \in C$ and $x_0 \in \mathrm{Sup}(C)$, then $x_0 \leq f(x_0)$.

If (S3) is fulfilled, then f is said to be *isotone* or *increasing*. Note also that (S3) implies (S4).

In the following, 2^X denotes the power set of X. For a multivalued function $F : X \to 2^X$, consider the following conditions mostly appeared in Smithson [9, 10].

(M1) for all $x \in X$, $S(x) \cap F(x) \neq \emptyset$, that is, there is a $y \in F(x)$ such that $x \leq y$.

(M2) there is an $e \in X$ such that $S(e) \cap F(e) \neq \emptyset$.

(M3) if $x_1 \leq x_2$ and $y_1 \in F(x_1)$, then there is a $y_2 \in F(x_2)$ such that $y_1 \leq y_2$.

(M4) if C is a well-ordered subset of X such that $x + 1 \in F(x)$ for all $x \in C$ and $x_0 \in \mathrm{Sup}C$, then $S(x_0) \cap F(x_0) \neq \emptyset$.

(M5) if C is a well-ordered subset of X such that there is an increasing selection $f(x) \in F(x)$ for all $x \in C$ and if $x_0 \in \mathrm{Sup}(C)$, then there is a $y_0 \in F(x_0)$ such that $f(x) \leq y_0$ for all $x \in C$.

Note that if $F = f$ is a single valued function, then (Mi) becomes (Si) for i = 1, 2, 3, 4 and that (M5) implies (M4).

We say that F is *isotone* (see Walker [13]) if F satisfies (M3) together with

(M3)' if $x_1 \leq x_2$ and $y_2 \in F(x_2)$, then there is a $y_1 \in F(x_1)$ such that $y_1 \leq y_2$.

The following is a slight extension of Smarzewski's lemma [8].

Lemma 1. *Let $(X, \leq)$ be a complete preordered set and $F : X \to 2^X$ satisfy (M1). Then there exists an order extremal subset $M \subset$ Max(X) such that $F(x) \cap M \neq \emptyset$ for all $x \in M$.*

Proof. Let $Y = X/\sim$. Since X is complete, Y is a complete partially ordered set. By Zorn's lemma, Y has a maximal element $M = [x_0]$, for some $x_0 \in X$. As a subset of X, $M \subset \mathrm{Max}(X)$ and is order extremal. Moreover, if $x \in M$, there exists some $y \in F(x)$ such that $x \in y$. By the extremality of M, $y \in M$. So $F(x) \cap M$ is nonempty.

Using Lemma 1, we characterize the fixed point property for multivalued functions as follows;

Theorem 1. *Let X be a nonempty set and $F : X \to 2^X \setminus \{\emptyset\}$ a multivalued function. Then F has a fixed point if and only if there is a preorder $\leq$ on X such that X has normal order structure and F satisfies (M1).*

Proof. Suppose that there is a preorder $\leq$ on X such that X has normal order structure and F satisfies (M1). By Lemma 1, there exists an order extremal subset M of X such that $F(x) \cap M \neq \emptyset$ for all $x \in M$. Since M is a singleton, let $M = \{x_0\}$. Then $x_0 \in F(x_0)$.

Conversely, suppose that F has a fixed point in X. Define $f : X \to X$ by

$$f(x) = \begin{cases} x, & \text{if } x \in F(x) \\ \text{any element of } F(x), & \text{if } x \notin F(x). \end{cases}$$

Then f is a single valued function with fixed point. By Smarzewski's theorem [8], there exists a preorder $\leq$ on X such that X has normal order structure and for all $x \in X$, $x \leq f(x)$. Since $f(x) \in F(x)$, the proof is complete.

A simple observation of Smarzewski's proof enables us to obtain a characterization of fixed point property as follows;

Theorem 2. *Let X be a nonempty set and $f : X \to X$ a function. Then f has a fixed point if and only if there is a preorder $\leq$ on X such that X has normal order structure and f satisfies (S2) and (S3).*

Note that Abian and Brown [1] proved a fixed point theorem for functions satisfying (S2) and (S3) in complete partially ordered sets.

3. Fixed Points and the Completeness of Posets

In [12], Tasković proved that a partially ordered set $(X, \leq)$ is inductive if and only if every function $f : X \to X$ satisfying

(S1)' for all $x \in \text{Sub } f(X)$, $x \leq f(x)$

has a fixed point, where

$$\text{Sub}f(X) = f(X) \cup \{x \mid x \text{ is an upper bound of some chain in } f(X)\}$$

For multivalued function $F : X \to 2^X$ we also define

$$\text{Sub}F(X) = F(X) \cup \{x \mid x \text{ is an upper bound of some chain in } F(X)\}$$

Obviously, F maps Sub$F(X)$ into itself. Moreover, it is easy to show that if X is inductive, so is Sub$F(X)$. Therefore, if we consider F as a function from Sub$F(X)$ into itself, Tasković's result is equivalent to the following;

Theorem 3. *Let $(X, \leq)$ be a partially ordered set. Then X is inductive if and only if every multivalued function $F : X \to 2^X$ satisfying (M1) has a fixed point.*

Proof. If X is inductive, then X has a maximal element x_0 by Zorn's lemma. By (M1), there is a $y \in F(x_0)$ such that $x_0 \leq y$. Since x_0 is maximal, $x_0 = y \in F(x_0)$.

Conversely, suppose that every multivalued function $F : X \to 2^X$ satisfying (M1) has a fixed point. Then every single valued function $f : X \to X$ satisfying (S1) has a fixed point. So by Tasković's theorem, X is inductive.

If C is a well-ordered subset of a preordered set X and $x \in C$, then $x + 1$ will denote the immediate successor of x, if exists. Let $\mathcal{C}$ be a set of well-ordered subsets of X and define a relation $\preccurlyeq$ on $\mathcal{C}$ by

$$C \preccurlyeq D \Longleftrightarrow C = D \text{ or } C \text{ is an initial segment of } D.$$

It is easy to show that $\preccurlyeq$ is a partial order on $\mathcal{C}$.

Theorem 4. *Let X be a preordered set and $F : X \to 2^X$ be a multivalued function. Suppose that (M2) and (M3) hold. Then either*

(a) F has a fixed point, or

(b) the set $\mathcal{C} = \{C \subset X \mid C$ is well-ordered and $x + 1 \in F(x)$ for all $x \in C\}$ is nonempty and has a maximal element with respect to $\preccurlyeq$.

Proof. Suppose that F has no fixed point. By (M2), there is an element $e_1 \in X$ such that $S(e_1) \cap F(e_1) \neq \emptyset$. That is, there is an element $e_2 \in F(e_1)$ such that $e_1 \leq e_2$. Note that $e_1 \neq e_2$, since F has no fixed point. Assume that $e_1, e_2, \cdots, e_n$ were chosen so that $e_i \leq e_{i+1}$ and $e_{i+1} \in F(e_i)$ for $i = 1, 2, \cdots, n-1$. Then (M3) shows that there is an $e_{n+1} \in F(e_n)$ such that $e_n \leq e_{n+1}$. By induction, we can construct a well-ordered set $C = \{e_n\}$ such that $e_n + 1 = e_{n+1} \in F(e_n)$ for all $e_n \in C$. Thus $C \in \mathcal{C}$ and $\mathcal{C} \neq \emptyset$.

Let $\mathcal{F}$ be a nonempty well-ordered subset of $\mathcal{C}$ with respect to $\preccurlyeq$. We will show that $\bigcup \mathcal{F}$ is an upper bound of $\mathcal{F}$. Let A be a nonempty subset of $\bigcup \mathcal{F}$. Since $\mathcal{F}$ is well-ordered, the set $\{C \in \mathcal{F} \mid A \cap C \neq \emptyset\}$ has the first element C_0. And since $A \cap C_0$ is a nonempty subset of a well-ordered set C_0, it also has the first element

x_0. Let $x \in A$ be arbitrary and choose $C \in \mathcal{F}$ such that $x \in C$. By the definition of C_0, $C_0 \preccurlyeq C$. Hence $C_0 = C$ or C_0 is an initial segment of C. Then x_0 is the first element of $A \cap C$. Thus $x_0 \leq x$ and so x_0 is the first element of A. This shows that $\bigcup \mathcal{F}$ is well-ordered. Moreover, if $x \in \bigcup \mathcal{F}$, then $x \in C$ for some $C \in \mathcal{F}$. Then $x+1 \in F(x)$ and thus $\bigcup \mathcal{F} \in \mathcal{C}$. So $\bigcup \mathcal{F}$ is an upper bound of $\mathcal{F}$. By Zorn's lemma, $\mathcal{C}$ has a maximal element. This completes the proof.

Corollary. *Let X be a preordered set and $f : X \to X$ be a self-function satisfying (S2) and (S3). Then either*

(a) f has a fixed point, or

(b) the set $\mathcal{C} = \{C \subset X \mid C$ is well-ordered and $f(x) = x + 1$ for all $x \in C\}$ is nonempty and has a maximal element with respect to $\preccurlyeq$.

Note that every element of $\mathcal{C}$ in Theorem 4 or in the above Corollary is an infinite set. So if X has no infinite well-ordered subset, then we have;

Theorem 5. *Let X be a preordered set having no infinite well-ordered subset. Suppose that $F : X \to 2^X$ is a multivalued function satisfying (M2) and (M3). Then F has a fixed point.*

Theorem 5 improves the result of Walker [13, Proposition 5. 2]. Furthermore, if X satisfies (D), we obtain the following extension of Smithson's theorem [9, Theorem 1. 1].

Theorem 6. *Let X be a preordered set satisfying (D). Let $F : X \to 2^X$ be a multivalued function such that (M2), (M3) and (M4) hold. Then F has a fixed point.*

Proof. Suppose that F has no fixed point. Theorem 4 shows that the set $\mathcal{C}$ has a maximal element C_0 with respect to $\preccurlyeq$. Then $x_0 \in \mathrm{Sup}(C_0)$ exists. By (M4), there is an $x_1 \in F(x_0)$ such that $x_0 \leq x_1$. As in the proof of Theorem 4, we can construct a well-ordered sequence $C = \{x_n\}$ such that $x_n \leq x_{n+1}$ and $x_{n+1} \in F(x_n)$ for all $n = 0, 1, \cdots$. Then $C_0 \cup C \in \mathcal{C}$ and $C_0 \preccurlyeq C_0 \cup C$, which contradicts the maximality of C_0.

Since (S3) implies (S4), if $F = f$ is a single valued function, theorem 6 reduces to the theorem given by Abian and Brown [1];

Corollary [1]. *Let X be a preordered set satisfying (D). Let $f : X \to X$ be a function such that (S2) and (S3) hold. Then f has a fixed point.*

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