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A PROOF OF THE MOST IMPORTANT IDENTITY INVOLVED IN THE BETA FUNCTION

JUNESANG CHOI

ABSTRACT. A new proof of the well-known identity involved in the Beta function $B(p,q)$ is given by using the theory of hypergeometric series and a brief history of Gamma function is also provided. The method here is shown to be able to apply to evaluate some definite integrals.

1. Introduction

The birth of the Gamma function was seen in two letters from Leonhard Euler (1707-1783) to Christian Goldbach (1690-1764), just as the simple desire to extend factorials to values between the integers. The first letter dated October 13, 1729 dealt with the interpolation problem, while the second dated January 8, 1730 dealt with integration and tied the two together. Euler gave us the well-known Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \qquad \mathrm{Re}(z) > 0, \tag{1}$$

where the notation Γ is due, in fact, to Adrin Marie Legendre (1752-1833).

Euler considered z as the positive real numbers in (1) and the move to the complex plane was initiated by Carl Friedrich Gauss (1777-1855). Legendre calls the integral (1) the second Eulerian integral. The first Eulerian integral is currently known as the Beta function and is now conventionally written

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, dt, \tag{2}$$

where $\mathrm{Re}(p) > 0$ and $\mathrm{Re}(q) > 0$.

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The Gamma function satisfies the relationships:

$$\Gamma(z+1)=z\Gamma(z), \qquad \Gamma(n+1)=n!\ (n=0,1,2,\ldots), \qquad \text{and}\ \ \Gamma(1/2)=\sqrt{\pi}. \tag{3}$$

There is the well-known relationship between Euler's two types of integrals (see Choi and Nam [2]):

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad \text{and so} \qquad B(p,q)=B(q,p). \tag{4}$$

For an arbitrary (real or complex) parameter α, define a binomial coefficient by

$$\binom{\alpha}{0}=1 \ \text{and}\ \binom{\alpha}{n}=\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \quad (n=1,2,3,\ldots) \tag{5}$$

so that

$$\binom{-\alpha}{n}=\frac{(-1)^n(\alpha)_n}{n!}=\frac{(-1)^n}{n!}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \quad (n=0,1,2,\ldots) \tag{6}$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the *generalized factorial*, since $(1)_n=n!$) defined by

$$(\alpha)_0=1 \qquad \text{and} \qquad (\alpha)_n=\alpha(\alpha+1)\ldots(\alpha+n-1) \ \ (n=1,2,3,\ldots). \tag{7}$$

The familiar binomial expansion is given as follows:

$$(1+z)^\alpha=\sum_{n=0}^{\infty}\binom{\alpha}{n}z^n \tag{8}$$

$$(|z|<1;\ \ \mathrm{Re}(\alpha)>0 \ \text{if}\ z=-1;\ \ \alpha=N \ \text{if}\ z=1)$$

N being a non-negative integer.

The well-known hypergeometric series is defined by

$$\begin{aligned} {}_2F_1(a,b;\,c;\,z) &= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!} \end{aligned} \tag{9}$$

$$(|z|<1;\ c\neq 0,-1,-2,\ldots;\ \ \mathrm{Re}(c-a-b)>0 \ \text{if}\ z=1;\ \ \mathrm{Re}(c-a-b)>-1 \ \text{if}\ z=-1)$$

which, for $a = c$ and $b = 1$ (or, alternatively, for $a = 1$ and $b = c$), reduces immediately to the relatively more familiar geometric series. In fact, in his 1812 thesis [3], Gauss proved his famous summation theorem:

$$ {}_2F_1(a, b;\, c;\, 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \mathrm{Re}(c-a-b) > 0 \ (c \neq 0, -1, -2, \ldots) \tag{10} $$

or, equivalently,

$$ \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \mathrm{Re}(c-a-b) > 0 \ (c \neq 0, -1, -2, \ldots). \tag{11} $$

Note that the hypergeometric series ${}_2F_1(a, b;\, c;\, z)$ is a solution of the following linear differential equation of the second order

$$ z(1-z)\frac{d^2w}{dz^2} + \{c - (a+b+1)z\}\frac{dw}{dz} - ab\,w = 0, $$

which was given by Gauss. Thereafter lots of theories and formulae involving ${}_2F_1$ itself have been developed. Some of important polynomials in the theory of special functions, i.e., Bessel functions, Legendre polynomials, Hermite polynomials, Laguerre polynomials, etc., are expressed as the ${}_2F_1$.

As also noted in Whittaker *et al.* [6, p. 281], the name '*hypergeometric series*' was given by Wallis in 1655 to the series whose nth term is

$$ a\,\{a+b\}\,\{a+2b\}\,\cdots\,\{a+(n-1)b\}. $$

Euler used the term hypergeometric in this sense, the modern use of the term being apparently due to Kummer, *Journal für Math.* xv. (1836).

In the present note we are aiming at providing another proof of the most important identity (4) using the theory of hypergeometric series. Some definite integrals are also shown to be evaluated in the method here.

We first introduce one of the methods of proving the identity (4). For another proof see Rainville [5, pp. 18-19]. Indeed, setting $t = \tau/(1+\tau)$ in (2) yields

$$ B(p, q) = \int_0^{\infty} \frac{\tau^{p-1}}{(1+\tau)^{p+q}}\, d\tau. \tag{12} $$

Then, for $\mathrm{Re}(p) > 0$ and $\mathrm{Re}(q) > 0$, we have

$$\begin{aligned}
\Gamma(p)\Gamma(q) &= \int_0^\infty e^{-t} t^{q-1}\, dt \int_0^\infty e^{-\tau} \tau^{p-1}\, d\tau \\
&= \int_0^\infty e^{-t} t^{p-1}\, dt \left(t^q \int_0^\infty e^{-tx} x^{q-1}\, dx \right) \quad (\tau = tx) \\
&= \int_0^\infty x^{q-1}\, dx \int_0^\infty e^{-t(x+1)} t^{p+q-1}\, dt \\
&= \int_0^\infty \frac{x^{q-1}}{(1+x)^{p+q}}\, dx \int_0^\infty e^{-\tau} \tau^{p+q-1}\, d\tau \quad (t(x+1) = \tau) \\
&= B(p,q)\Gamma(p+q),
\end{aligned}$$

where the identity (12) is used for the last equality.

Now we give another proof of the identity (4). Using the binomial expansion (8), we obtain, for $\mathrm{Re}(p) > 0$ and $\mathrm{Re}(q) > 0$,

$$\begin{aligned}
B(p,q) &= \int_0^1 t^{p-1}(1-t)^{q-1}\, dt \\
&= \int_0^1 t^{p-1} \sum_{n=0}^\infty \binom{q-1}{n} (-t)^n\, dt \\
&= \sum_{n=0}^\infty (-1)^n \binom{q-1}{n} \int_0^1 t^{p+n-1}\, dt \\
&= \sum_{n=0}^\infty (-1)^n \binom{q-1}{n} \frac{1}{p+n},
\end{aligned}$$

where, for the third equality, we can readily justify that the order of integration and summation is exchangeable.

In view of the identity (6) and the fundamental functional relation (3), we find that

$$\begin{aligned}
B(p,q) &= \sum_{n=0}^\infty (-1)^n \binom{q-1}{n} \frac{1}{p+n} \\
&= \sum_{n=0}^\infty \frac{(1-q)_n}{n!} \frac{1}{p+n} \\
&= \sum_{n=0}^\infty \frac{(1-q)_n}{n!} \frac{\Gamma(p+n)}{\Gamma(p+n+1)} \\
&= \frac{\Gamma(p)}{\Gamma(p+1)} \frac{\Gamma(p+1)}{\Gamma(p)\Gamma(1-q)} \sum_{n=0}^\infty \frac{\Gamma(1-q+n)\Gamma(p+n)}{\Gamma(p+1+n)},
\end{aligned}$$

from which, in terms of the second definition of ${}_2F_1$ in (9), we obtain

$$\begin{aligned} B(p,q) &= \frac{\Gamma(p)}{\Gamma(p+1)}\,{}_2F_1(1-q,p;p+1;1) \\ &= \frac{\Gamma(p)}{\Gamma(p+1)}\frac{\Gamma(p+1)\Gamma(q)}{\Gamma(p+q)\Gamma(1)} \\ &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \end{aligned}$$

which is the desired identity (4), and for the second equality the Gauss's summation theorem (10) is used.

We conclude this note by remarking that the Gauss's summation theorem may seem to have lots of applications in evaluations of some types of definite integrals as well as in the theory of hypergeometric series itself.

As an illustration, we consider the following integral:

$$\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{1}{2\pi\sqrt{3}\sqrt[3]{2}}\left\{\Gamma\left(\frac{1}{3}\right)\right\}^3, \tag{13}$$

which was recorded in Gradshteyn *et al.* [4, p. 229, Entry 3.139]. Indeed,

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{1-x^3}} &= \int_0^1 \sum_{n=0}^{\infty}(-1)^n\binom{-1/2}{n}x^{3n}\,dx \\ &= \sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{n!}\frac{1}{3n+1} \\ &= \sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{3}\right)_n}{\left(\frac{4}{3}\right)_n n!} = {}_2F_1\left(\frac{1}{2},\frac{1}{3};\frac{4}{3};1\right) \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{6}\right)}. \end{aligned} \tag{14}$$

Now, recalling the well-known reflection and duplication formulae:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z} \quad \text{and} \quad \sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2), \tag{15}$$

we readily obtain

$$\Gamma\left(\frac{5}{6}\right) = \frac{2^{\frac{4}{3}}\pi^{\frac{3}{2}}}{\sqrt{3}}\left\{\frac{1}{\Gamma\left(\frac{1}{3}\right)}\right\}^2. \tag{16}$$

Finally combining (14) and (16) with the aid of (3) leads at once to (13). Similarly we may obtain

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2\pi}} \left\{\Gamma\left(\frac{1}{4}\right)\right\}^2; \tag{17}$$

$$\int_0^1 \frac{x\,dx}{\sqrt{1-x^3}} = \frac{1}{\pi}\frac{\sqrt{3}}{\sqrt[3]{4}} \left\{\Gamma\left(\frac{2}{3}\right)\right\}^3. \tag{18}$$

For another interesting application, consider the problem which was posed by Ananthanarayana Sastri [1, p. 80, No. 644]:

The length of the fourth positive pedal of a loop of the Lemniscate of Bernoulli is given by

$$18\,a \int_0^1 \frac{x^8\,dx}{\sqrt{1-x^4}},$$

where a denotes one half of the distance between two fixed points in the definition of the Lemniscate of Bernoulli.

Similarly as above, we evaluate this integral

$$18\,a \int_0^1 \frac{x^8\,dx}{\sqrt{1-x^4}} = \frac{15\sqrt{2}}{28} a \left\{\Gamma\left(\frac{1}{4}\right)\right\}^2, \tag{19}$$

the numerical value of $\Gamma(1/4)$ being $3.625\,609\,908\,221\cdots$.

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Department of Mathematics, College of Natural Sciences, Dongguk University, Kyongju 780-714, Korea