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SOME REMARKS ON UNIVERSAL PETTIS INTEGRAL PROPERTY

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ABSTRACT. Some function of a complete finite measure space (for short, CFMS) into the duals and pre-duals of weakly compactly generated (for short, WCG) spaces are considered. We shall show that a universally weakly measurable function f of a CFMS into the dual of a WCG space has RS property and bounded weakly measurable functions of a CFMS into the pre-duals of WCG spaces are always Pettis integrable.

1. Introduction

The notion of the so-called unversal Pettis integral property of Banach spaces (UPIP) was introduced by L. H. Riddle, E. Saab and J. J. Uhl (1983) in [9]. A Banach space X has the UPIP if for every compact Hausdorff space Ω, every universally weakly measurable function $f : \Omega \to X$ is universally Pettis integrable. In [9, Theorem 6] it was shown that the duals of separable spaces have the UPIP. The proof in [9] reveals the following property of universally weakly measurable functions:

$$\text{RS(D)}: \left\{ \begin{array}{l} \text{For every Radon measure } \mu \text{ on } \Omega \text{ and every separable subspace } D \\ \text{of } X^*, \text{ the family } Z_f(D) = \{x^*f : x^* \in D, \quad \| x^* \| \leq 1\} \text{ is almost} \\ \text{weakly pre-compact in } L_\infty(\mu). \end{array} \right.$$

If RS(D) holds for all subspaces D of X^*, whether it is separable or not, then f is said to have the RS property for μ. Here, RS stands for Riddle-Saab, a terminology is used by M.Talagrand(1984) in [12], where he asks if bounded universally weakly measurable functions always have the RS property.

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A key step in the proof of [9, Theorem 6] is to show that unversally weakly measurable functions of a CFMS into the duals of separable Banach spaces do have the RS property. In this paper we will show that a universally weakly measurable function of a CFMS into the duals of WCG spaces always has the RS property. Furthermore, it is shown that for any given Radon measure μ, any such function f can be written as a sum of two universally weakly measurable functions, one being universally Pettis integrable, the other μ-weak* equivalent to zero.

It is still an open question whether a dual of a WCG space X has the UPIP, but this decomposition shows that in order to determine which it is, it suffices to consider those functions that vanish almost everywhere against elements from X; that is, those for which $fx = 0$ almost everywhere.

The UPIP is a restriction of the more general Pettis integral property (PIP). A Banach space X is said to have the λ -PIP if for every finite and complete measure space $(\Omega, \Sigma, \lambda)$, every bounded weakly measurable function $f : \Omega \to X$ is Pettis integrable. We will show that pre-duals of WCG spaces have the PIP.

2. DEFINITIONS AND PRELIMINARIES

We present some necessary notations and terminologies which are needed in our subsequent section. Insofar as possible, we adopt the definitions and notations of [4] and [5]. Throughout this paper, $(\Omega, \Sigma, \lambda)$ will always be complete finite measure space, and the letters X, Y, Z will denote real Babach spaces with duals X^*, Y^*, Z^*, respectively. And the closed unit ball in X will be denoted by B_X.

Definition 2-1. (a) A subset K of a Banach space X is called *weakly pre-compact* if every sequence in K has a weak Cauchy subsequence.

(b) A bounded function $f : \Omega \to X$ (resp. $f : \Omega \to X^*$) is called *weakly measurable* (resp. *weak* measurable*) if for all x^* in X^* (resp. for all x in X) the scalar valued function x^*f (resp. xf) is measurable. A function $f : \Omega \to X(Y^*)$ is said to be μ-weakly zero (resp. μ-weak* zero) if x^*f (resp. yf) is zero λ -a.e.. Two functions f and g are called weakly equivalent (resp. weak* equivalent) if x^*f is equivalent to x^*g (resp. xf is equivalent to xg). A function $f : \Omega \to X$ is said to be determined by a subsequence D of X if for every $x^* \in X^*$, $x^*|_D = 0$ implies that $x^*f = 0$ λ-a.e..

Definition 2-2. (a) A weakly measurable function $f : \Omega \to X$ is said to be *Dunford integrable* if $x^*f \in L_1(\lambda)$ for all $x^* \in X^*$. The Dunford integral f of over $E \in \Sigma$ is defined by the element $x_E^{**} \in X^{**}$ such that $x_E^{**}(x^*) = \int_E x^* f d\lambda$ for all $x^* \in X^*$, and denote it by $x_E^{**} = (D) - \int_E f d\lambda$. In the case, that $(D) - \int_E f d\lambda$ belongs to X for each $E \in \Sigma$, then f is called λ-*Pettis integrable* and we write $(P) - \int_E f d\lambda$ instead of $(D) - \int_E f d\lambda$ to denote the *Pettis integral* of f over $E \in \Sigma$.

(b) The *weak* integral* of $f : \Omega \to X^*$ over E, denoted by $(w^*) - \int_E f d\lambda$, is the element x_E^* of X^* defined by the equation $x_E^*(x) = \int_E x f d\lambda$ for all $x \in X$.

Definition 2-3. A function f defined on a compact Hausdorff space Ω with values in a Banach space X is said to be *universally weakly measurable* if for every Radon measure μ on Ω and each x^* in X^*, the scalar valued function x^*f is μ-measurable. In this case, f is *universally Pettis integrable* if it is μ-Pettis integrable for each Radon measure μ on Ω .

3. MAIN RESULTS

The following two lemmas will be needed in order to ensure universal Pettis integrability of a given universally weakly measurable function $f : \Omega \to X^*$. For the proof, see [11].

Lemma 3-1. *Let $(\Omega, \Sigma, \lambda)$ be a complete finite measure space and let $f : \Omega \to X^*$ be weak* integrable, and assume that the set $\{xf : \| x \| \leq 1\} \subset L_1(\lambda)$ is separable. Then f is weak* equivalent to weak* measurable function g that takes its range in a weak* separable subspace of X^*.*

The second lemma is based on a result of [3], where it is shown that a WCG space has a one-to-one continuous image of a reflexive space as dense subspace.

Lemma 3-2. *Let $(\Omega, \Sigma, \lambda)$ be a complete finite measure space. If X^* is a dual of a WCG space X and $f : \Omega \to X^*$ is weak* integrable, then the set $\{xf : \| x \| \leq 1\} \subset L_1(\lambda)$ is separable.*

The above two lemmas together with the proof of [9, Theorem 6] allows us to prove the following :

Theorem 3-3. *Let X^* be a dual of a WCG space X and let $f : \Omega \to X^*$ be universally weakly measurable. For any Radon measure μ on Ω, the family $\{xf : \| x \| \leq 1\}$ is almost weakly pre-compact in $L_\infty(\mu)$. Furthermore, f can be written as a sum*

$$f = f_1 + f_2$$

where f_1 is universally Pettis integrable and f_2 is μ-weak equivalent to zero.*

Proof. Let μ be any Radon measure on Ω . By the above lemmas, f is weak* equivalent to weak* measurable function $g : \Omega \to X^*$ that takes its range in a weak* separable subspaces of X^*. By a theorem of Admir and Lindenstrauss [1], we can write X as a direct sum $X = X_1 \oplus X_2$, where X_2 is separable and g takes its range in X_2^*. Since X_2 is separable, there exists a set $E \in \Sigma$ with $\mu(E) = 0$ such that

$$x_2 f(w) = x_2 g(w), \text{ for all } x_2 \in X_2 \text{ and } w \in \Omega \setminus E.$$

Consequently $x_2^{**} f = x_2^{**} g$ a.e. for all $x_2^{**} \in X_2^{**}$ and hence, g is weakly measurable.

Write f as a sum $f = f_1 + f_2$, where f_i takes its range in X_i^*, $i = 1, 2$. It is clear that both f_1 and f_2 are universally weakly measurable. Since f_2 is valued in a dual of a separable space, it is universally Pettis integrable [9], and since for any x in X,

$$\begin{aligned} xf &= xg \quad \mu\text{-a.e.} \\ &= xf_2, \end{aligned}$$

it follows that f_2 is weak* equivalent to zero. The two sets

$$Z_f(B_X) = \{xf : x \in X, \| x \| \leq 1\} \text{ and } Z_g(B_X) = \{x_2 g : x_2 \in X_2, \| x_2 \| \leq 1\}$$

are equal as subsets of $L_\infty(\mu)$, and since $x_2 f$ is valued in the dual of a separable space, the set $Z_g(B_X)$, and thus $Z_f(B_X)$ is almost weakly pre- compact in $L_\infty(\mu)$. Since the measure μ was an arbitrary Radon measure on Ω, the proof is complete. □

Remark 1. In [2] Bator shows that universally weakly measurable functions are Pettis decomposable ; that is, for any Radon measure μ, any universally weakly measurable function f can be written as a sum $f = f_1 + f_2$, where f_2 is μ-Pettis integrable, and f_1 is μ-weak* equivalent to zero. This decomposition can be done regardless of the Banach spaces in which the functions take their range. However, the functions f_1 and f_2 need not be universally weakly measurable, nor need either function be integrable with respect to other Radon measures.

Corollary 3-4. *Let X^* be a dual of a WCG space. Then the following ststements are eqvivalent :*

(i) X^* has the UPIP.

(ii) Let Ω be any compact Hausdorff space and let μ be any Radon measure on Ω . For any universally weakly measurable function $f : \Omega \to X^*$, the set $Z_f(B_X)$ is dense in $Z_f(B_{X^{**}})$.

(iii) Let Ω be a compact Hausdorff space, and let μ be any Radon measure on Ω . Every universally weakly measurable function $f : \Omega \to X^*$ which is μ-weak zero is weakly zero.*

Proof. (i)$\Rightarrow$ (ii)$\Rightarrow$ (iii) is clear. (iii)$\Rightarrow$ (i) follows from Theorem 3-3. □

Theorem 3-5. *If X has a dual of a WCG space, then X has the PIP.*

Proof. Let $(\Omega, \Sigma, \lambda)$ be any complete finite measure space and let $f : \Omega \to X$ be bounded and weakly measurable. We want to show that f is Pettis integrable. In order to do it, we show that f is determined by a separable subspace of X, and the result follows from [11].

View f as a function into X^{**}. By Lemma 3-1 and 3-2, f is weak* equivalent to weak* measurable function $g : \Omega \to X^{**}$ that takes its range in a weak* separable subspace Y of X^{**}. This means that whenever $x^*|_Y = 0$ $x^*g(w) = 0$ for almost all w, and hence $x^*f = 0$ almost everywhere. Let $\{y_n : n \in N, \| y_n \| \leq 1\}$ be a weak* dense subset of the unit ball of Y, and for each n, find a bounded sequence (x_{nk}) in X such that $x_{nk} \to y_n$ weak*. If D is the span of $\{x_{nk} : n, k \in N\}$, then D is a separable subspace of X and f is determined by D. Indeed, if $x^*|_D = 0$, then $x^*(x_{nk}) = 0$ for all n and k, and hence $y_n(x^*) = 0$ for all n. Consequently, $x^*|_Y = 0$ and thus $x^*f = 0$ almost everywhere. □

Remark 2. Since X^* is WCG, it has the Radon-Nikodim property(RNP). In particular, X fails to contain a copy of ℓ_1 . It follows that not only X does have the PIP, but also the indefinite integral of every Pettis integrable function $f : \Omega \to X$ has a relatively compact range. This holds true whether the function is bounded or not. This is, in certain sense, the closest functions into preduals of WCG spaces come to imitate strongly measurable functions. For example, all duals of the James Tree space have the PIP and the weak RNP. However, only the even duals (The WCG ones) have the RNP.

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