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TWO THEOREMS FOR POISSON MEASURES ON HYPERGROUPS

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ABSTRACT. Our first theorem is concerned with the convergence of nets of Poisson measures on a hypergroup. As a corollary we obtain a characterization of Poisson measures. The second theorem gives a characterization of elementary Poisson measures.

I. Introduction

Hypergroups were introduced independently by C. F. Dunkl, R. I. Jewett and R. Spector, and successfully pursued by K. A. Ross, A. K. Chilana, H. Heyer, W. R. Bloom , R. Lasser and others. There are many papers that are concerned with the characterization of Poisson measures on locally compact groups, but not on hypergroups. Our main theorems are concerned with the convergence of nets and the necessary and sufficient condition for Poisson measures on hypergroups. Let G be a locally compact group. By $\mathfrak{M}^b(G)$ we denote the space of bounded Radon measures on G furnished with the weak topology $\mathcal{T}_w$, and we denote by $\mathfrak{M}^1(G)$ the space of probability measures on G. Then the space $\mathfrak{M}^1(G)$ is a Banach algebra under the total variation norm. The Dirac measure in a point $x \in G$ will be abbreviated by ε_x.

A hypergroup G is said to be a *commutative* if $\varepsilon_x * \varepsilon_y = \varepsilon_y * \varepsilon_x$ for any x, y in G, where the notation "$*$" denotes the convolution in G. Our definitions on a hypergroup are in the sense of R. I. Jewett[4]. Throughout in this paper, we assume that G is a commutative and compact hypergroup. By $\mathcal{C}^b(G)$ we denote the space of real valued bounded continuous functions on G equipped with the supremum norm $\|\cdot\|_\infty$. For any compact subgroup H of G, we shall denote the normed Haar measure

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of H by ω_H. Then the class $\mathfrak{M}^b_H(G) := \omega_H * \mathfrak{M}^b(G) * \omega_H$ is a Banach subalgebra of $\mathfrak{M}^b(G)$ with the Harr measure ω_H as multiplicative unit with respect to convolution $*$ and the norm $\|\cdot\|$ of total variation. We can introduce the *exponential* $\exp_H(\nu)$ of measure $\nu \in \mathfrak{M}^b_+(G)$ by

$$\exp_H(\nu) = \omega_H + \sum_{k\geq 1} \frac{\nu^k}{k!},$$

where $\mathfrak{M}^b_+(G)$ is the subset of positive measures in $\mathfrak{M}^b(G)$. If $\nu \in \mathfrak{M}^b_+(G)$ satisfies that for any compact subgroup H of G

$$\omega_H * \nu * \omega_H = \nu,$$

we define an *H-Poisson measure* μ with exponent ν as following :

$$\mu = \exp_H(\nu - \|\nu\|\omega_H),$$

or

$$\mu = \exp_H(\lambda) = \omega_H + \sum_{k\geq 1} \frac{\lambda^k}{k!}$$

where $\lambda = \nu - \|\nu\|\omega_H$. If $\mu_t = \exp_H(t\lambda)$ for all real number $t \geq 0$, then $(\mu_t)_{t\geq 0}$ is called a *H-Poisson semigroup* on G. Now by $\mathcal{P}_H(G)$ we will denote the set of all H-Poisson measures on G. If G is a commutative compact hypergroup, by $\widehat{G}$ we denote the character group of G. For any H-Poisson measure μ, its Fourier transform is denoted by $\mu^\wedge$. A measure $\mu \in \mathcal{P}_H(G)$ of the form

$$\mu = \exp_H[\gamma(\lambda - \omega_H)] \quad \text{with} \quad \lambda \in \mathfrak{M}^1_H(G) \quad \text{and} \quad \gamma > 0$$

is called *elementary* if there exists an $x_0 \in G$ satisfying $\lambda = \omega_H * \varepsilon_{x_0} * \omega_H$. In particular, a measures $\mu \in \mathfrak{M}^1(G)$, $\mu \neq \varepsilon_e$, is called an *elementary Poisson measure with parameter* $x_0 \in G$ if

$$\begin{aligned}\mu &= \exp_{\{e\}}[\gamma(\varepsilon_{x_0} - \varepsilon_e)] \\ &= e^{-\gamma}\left[\varepsilon_e + \gamma\varepsilon_{x_0} + \frac{\gamma^2}{2!}\varepsilon_{x_0^2} + \cdots\right]\end{aligned}$$

with some constant $\gamma > 0$.

II. Theorems

Suppose $(\nu_\alpha)_{\alpha\in I}$ is a net of totally finite measures such that $\omega_H * \nu_\alpha * \omega_H = \nu_\alpha$ for all $\alpha \in I$ where I is any index set. Let $\lambda_\alpha = \nu_\alpha - \|\nu_\alpha\|\omega_H$ for every $\alpha \in I$. Also we define

$$\mu_t^{(\alpha)} = \exp_H(t\lambda_\alpha)\,, \qquad t \geq 0$$

and

$$S_\alpha = \left(\mu_t^{(\alpha)}\right)_{t\geq 0}, \qquad S = (\mu_t)_{t\geq 0}\,.$$

Then S_α is an H-Poisson semigroup for each α. By $\mathbf{1}_{G\setminus H}\cdot\mu$ we denote the measure defined by

$$\left(\mathbf{1}_{G\setminus H}\cdot\mu\right)(B) = \mu(B) - \mu(B\cap H)$$

for all Borel set B and $\mu \in \mathfrak{M}_+^b(G)$. For any index set I, a net $(\mu_t)_{t\in I}$ is said to be a *tight net* if for every $\varepsilon > 0$ there exists a compact subset $K := K_\varepsilon$ of G and $t_0 := t_0(\varepsilon) \in I$ such that $\mu_t(K^c) < \varepsilon$ holds for all $t > t_0$ and $\overline{\lim}_{t\in I}\|\mu_t\| < \infty$.

Theorem 1. *Let the following two conditions be satisfied :*
(i) $\left(\mu_1^{(\alpha)}\right)_{\alpha\in I}$ *is a tight net in* $\mathfrak{M}^1(G)$.
(ii) $\overline{\lim}_{\alpha\in I}\ \nu_\alpha(G\setminus H) < \infty$.
Then (a) a net $\left(\mathbf{1}_{G\setminus H}\cdot\nu_\alpha\right)_{\alpha\in I}$ *is tight.*
(b) $(S_\alpha)_{\alpha\in I}$ *is compact in* $\mathcal{P}_H(G)$ *with respect to the weak topology. More if* ν *is a weak limit point of* $\left(\mathbf{1}_{G\setminus H}\cdot\nu_\alpha\right)_{\alpha\in I}$*, then the H-Poisson semigroup* $S = (\mu_t)_{t\geq 0}$ *is a limit point of* $(S_\alpha)_{\alpha\in I}$ *where* $\mu_t = \exp_H[t(\nu - \|\nu\|\omega_H)]$.

Lemma 1. *If* $\left(\mu_1^{(\alpha)}\right)_{\alpha\in I}$ *is a tight net, then* $\left(\mu_t^{(\alpha)}\right)_{\alpha\in I}$ *is also tight for* $t \geq 0$.

Proof. Since $\mu_t^{(\alpha)} \in \mathfrak{M}_+^b(G)$ for all $t \geq 0$ and $\alpha \in I$, without loss of generality we may assume that $\nu_a(G\setminus H) \leq c/2$ for all $\alpha \in I$ and for some constant c. Moreover we can assume $\nu_\alpha(H) = 0$, then $\mathbf{1}_{G\setminus H}\cdot\nu_\alpha = \nu_\alpha$. Let $0 < \varepsilon < 1/3$. Then there exists a compact subset $K := K_\varepsilon$ of G and an index $\alpha_0 := \alpha_0(\varepsilon) \in I$ such that $\mu_1^{(\alpha)}(K) > 1 - \varepsilon$ for $\alpha > \alpha_0$. Since H is compact we may assume $KH = K$. Given $0 < t < 1$ and $\alpha \in I$ we define

$$K_t^{(\alpha)} = \{x \in G \mid \mu_t^{(\alpha)}(xK) > 1 - \varepsilon\}\,.$$

Since

$$1-\varepsilon<\mu_1^{(\alpha)}(K)=\mu_{1-t}^{(\alpha)}*\mu_t^{(\alpha)}(K)$$
$$=\int_G \mu_t^{(\alpha)}(y^{-1}K)\mu_{1-t}^{(\alpha)}(dy)\,,$$

we have $K_t^{(\alpha)}\neq\phi$ for $\alpha>\alpha_0$. Furthermore since $\|\lambda_\alpha\|\leq 2\nu_\alpha(G\backslash H)\leq c$, we have

$$\|\mu_t^{(\alpha)}-\omega_H\|=\left\|\omega_H+\sum_{k=1}^{\infty}\frac{t^k}{k!}\lambda_\alpha^k-\omega_H\right\|$$
$$\leq\sum_{k=1}^{\infty}\frac{t^k}{k!}\|\lambda_\alpha\|^k$$
$$\leq\sum_{k=1}^{\infty}\frac{(tc)^k}{k!}\leq tc\cdot e^c \qquad (t\geq 0)\,.$$

Now we choose $0<t_0<1$ with $t_0ce^c<1/3$, and let $0<t<t_0$, $\alpha>\alpha_0$ and $x\in K_t^{(\alpha)}$. Then

$$\|\omega_H(xK)-\mu_t^{(\alpha)}(xK)\|\leq\|\omega_H-\mu_t^{(\alpha)}\|\leq tce^c<\frac{1}{3}\,,$$

and hence

$$\omega_H(xK)>\mu_t^{(\alpha)}(xK)-\frac{1}{3}>\frac{2}{3}-\varepsilon>0\,.$$

Thus we have $H\cap xK\neq\phi$ and $x\in K^{-1}$ since $KH=K$. This proves

$$\mu_t^{(\alpha)}(K^{-1}K)\geq\mu_t^{(\alpha)}(xK)>1-\varepsilon\,.$$

Hence $(\mu_t^{(\alpha)})_{\alpha\in I}$ is a tight net for any $0<t<t_0$ and for any $t\geq 0$ by Lemma 2.1[7]. ∎

Lemma 2. *Let the following two conditions be satisfied :*
(i) For any $t\geq 0$, $\left(\mu_t^{(\alpha)}\right)_{\alpha\in I}$ *is a tight net.*
(ii) $\overline{\lim}_{\alpha\in I}\,\nu_\alpha(G\backslash H)<\infty$.
Then $(\nu_\alpha)_{\alpha\in I}$ *is a tight net.*

Proof. Without loss of generality we may assume that $\nu_\alpha(H)=0$. Let $\varepsilon>0$ and c be a constant with $\|\lambda_\alpha\|\leq c$. Then there exists a real number $0<t_0<1$ such that $t_0e^c \le \varepsilon/2$. Since $\left(\mu_{t_0}^{(\alpha)}\right)_{\alpha\in I}$ is a tight net by condition (i). Hence there is a compact set $K := K_\varepsilon$ of G and an index $\alpha_0 := \alpha_0(\varepsilon) \in I$ such that

$$\mu_{t_0}^{(\alpha)}[(G\backslash H)\backslash K] \le \mu_{t_0}^{(\alpha)}(G\backslash K) \le \frac{1}{2}\varepsilon t_0 \quad \text{for } \alpha > \alpha_0.$$

Since $\|\lambda_\alpha\| \le c$, we have : for $0 < t < 1$,

$$\begin{aligned}\left\|\frac{1}{t}(\mu_t^{(\alpha)} - \omega_H) - \lambda_\alpha\right\| &= \left\|\frac{1}{t}\sum_{k=1}^{\infty}\frac{t^k}{k!}\lambda_\alpha^k - \lambda_\alpha\right\| \\ &= \left\|\sum_{k=2}^{\infty}\frac{t^{k-1}}{k!}\lambda_\alpha^k\right\| \\ &\le t\sum_{k=2}^{\infty}\frac{t^{k-2}}{k!}\|\lambda_\alpha\|^k \\ &= te^c\end{aligned}$$

Thus we have : for $\alpha > \alpha_0$,

$$\begin{aligned}\nu_\alpha(G\backslash K) &= \nu_\alpha[(G\backslash H)\backslash K] = \lambda_\alpha[(G\backslash H)\backslash K] \\ &\le \frac{1}{t_0}\left[\mu_{t_0}^{(\alpha)} - \omega_H\right]((G\backslash H)\backslash K) + t_0e^c \\ &\le \frac{1}{t_0}\mu_{t_0}^{(\alpha)}[(G\backslash H)\backslash K] + \frac{\varepsilon}{2} \le \varepsilon\,.\end{aligned}$$

Together with condition (ii), this proves our statement. ∎

Proof of theorem 1. We assume that I is a universal net. Then by Lemma 1 $\left(\mu_t^{(\alpha)}\right)_{\alpha\in I}$ is tight for $t \ge 0$. By Lemma 2 $(\mathbf{1}_{G\backslash K}\nu_\alpha)_\alpha$ is tight and there exists a measure $\nu \in \mathfrak{M}_+^b(G)$ such that $\mathcal{T}_w - \lim_\alpha \nu_\alpha = \nu$. By the weak continuity of convolution in $\mathfrak{M}_+^b(G)$ we have $\omega_H * \nu * \omega_H = \nu$. Let $\mu_t = \exp_H[t(\nu - \|\nu\|\omega_H)]$ for $t \ge 0$ and $S = (\mu_t)_{t\ge 0}$. Moreover if we can show $(S_\alpha)_\alpha$ converges to S weakly, then $(S_\alpha)_\alpha$ is compact by properties 1.2.20[2]. Now in order to show $\lim_\alpha S_\alpha = S$, let $d > 0$ and $\varepsilon > 0$. Since $\|\nu_\alpha\| \le \frac{c}{2}$ and $\|\nu\| \le \frac{c}{2}$ for some constant c, there exists an integer n such that $\left|\sum_{k\ge n}\frac{t^k}{k!}\nu^{\wedge k}(\chi)\right| \le \varepsilon$ and $\left|\sum_{k\ge n}\frac{t^k}{k!}\nu_\alpha^{\wedge k}(\chi)\right| \le \varepsilon$ for all $0 \le t \le d$ and $\chi \in \widehat{G}$. Then we have : for all $\alpha \in I$,

$$\begin{aligned}\left|\mu_t^{(\alpha)\wedge}(\chi) - \mu_t^{\wedge}(\chi)\right| &= \left|e^{-t\|\nu_\alpha\|}\sum_{k\ge 0}\frac{t^k}{k!}\nu_\alpha^{\wedge k}(\chi) - e^{-t\|\nu\|}\sum_{k\ge 0}\frac{t^k}{k!}\nu^{\wedge k}(\chi)\right| \\ &\le \sum_{k=0}^{n}\frac{t^k}{k!}\left|e^{-t\|\nu_\alpha\|}\nu_\alpha^{\wedge k}(\chi) - e^{-t\|\nu\|}\nu^{\wedge k}(\chi)\right| + 2\varepsilon\,.\end{aligned}$$

Since $\lim_\alpha e^{-t\|\nu_\alpha\|} = e^{-t\|\nu\|}$ uniformly in $t \in [0, d]$ and $\lim_\alpha \nu_\alpha^{\wedge k}(\chi) = \nu^{\wedge k}(\chi)$ by the weak continuity of convolution, we have :

$$\lim_\alpha \mu_t^{(\alpha)\wedge}(\chi) = \mu_t^\wedge(\chi) \quad \text{uniformly in} \quad t \in [0, d]$$

or,

$$\mathcal{T}_w - \lim_\alpha \mu_t^{(\alpha)} = \mu_t\,.$$

Hence the theorem is completely proved. ∎

Corollary 1. *Let G be a hypergroup and H a compact subgroup of G. Then the following statements are equivalent :*
(i) μ is an H-Poisson measure.
(ii) There exist a sequence $(k_n)_{n\geq 1}$ of integers and a measure $\mu_n \in \mathfrak{M}^1(G)$ such that for all integer n,

$$\omega_H * \mu_n * \omega_H = \mu_n\,, \quad \mu_n^{k_n} = \mu\,, \quad \overline{\lim}\; k_n\mu_n(G\backslash H) < \infty\,.$$

Proof. "(i) ⇒ (ii)" If $\mu \in \mathcal{P}_H(G)$, then there exists a measure $\nu \in \mathfrak{M}_+^b(G)$ such that $\omega_H * \nu * \omega_H = \nu$ and $\mu = \exp_H(\nu - \|\nu\|\omega_H)$. Let $k_n = n$ and $\mu_n = \exp_H\left(\frac{1}{n}(\nu - \|\nu\|\omega_H)\right)$. Then we, clearly, have : $\mu_n^n = \mu$ for all $n \geq 1$ and

$$\omega_H * \mu_n * \omega_H = \omega_H * \left(\omega_H + \sum_{k\geq 1}\frac{\lambda^k}{k!n^k}\right) * \omega_H$$

$$= \omega_H + \sum_{k\geq 1}\frac{\lambda^k}{k!n^k} = \mu_n \qquad \text{for each } n,$$

moreover

$$\lim_{n\to\infty}\|n(\mu_n - \omega_H) - \lambda\| = \lim_{n\to\infty}\sum_{k\geq 2}\frac{\lambda^k}{k!n^{k-1}} = 0\,.$$

Hence $\overline{\lim}_n\|\mu_n - \omega_H\| < \infty$. Since $\|\mu_n - \omega_H\| = 2\mu_n(G\backslash H)$, we obtain $\overline{\lim}_n\mu_n(G\backslash H) < \infty$.

"(ii) ⇒ (i)" Let $\nu_n = k_n\mu_n$ and for each $n = 1, 2, 3, \cdots$,

$$\eta_y^{(n)} = \exp_H\left(t(\nu_n - \|\nu_n\|\omega_H)\right) = \exp_H(tk_n(\mu_n - \omega_H))\,.$$

Then we have :

$$\omega_H * \nu_n * = \nu_n \qquad \text{and} \qquad \overline{\lim}_n\nu_n(G\backslash H) < \infty\,.$$

Then we can obtain $\lim_{n\to\infty}\|\mu - \eta_1^{(n)}\| = 0$(see in the proof of Lemma 1.11[6]). Thus $\left(\eta_1^{(n)}\right)_{n\geq 1}$ is tight and μ is the weak limit point of $\left(\eta_1^{(n)}\right)_{n\geq 1}$. That is, by Theorem 1 μ is an H-Poisson measure. ∎

Theorem 2. *Let G be a compact hypergroup, $\mu \in \mathfrak{M}^1(G)$ with $\mu \neq \varepsilon_e$ and $x_0 \in G$ satisfying $x_0^2 \neq e$. Then the following statements are equivalent :*

(i) μ is an elementary Poisson measure with parameter x_0.

(ii) For all $n \geq 1$ there exists an n-th root $\mu_n \in \mathfrak{M}^1(G)$ of μ such that

(a) $\overline{\lim}_n \mu_n(\{e\}) = 1$,

(b) $\lim_{n\to\infty} n\mu_n(G\backslash\{e, x_0\}) = 0$.

Proof. "(i) $\Rightarrow$ (ii) " Put $\mu_n = \exp_{\{e\}}\left[\frac{\gamma}{n}(\varepsilon_{x_0} - \varepsilon_e)\right]$. It is clear that $\mu = \mu_n^n$ and $\mu_n \in \mathfrak{M}^1(G)$ for all n. Since for each n

$$\begin{aligned}\mu_n(\{e\}) &= e^{-\gamma/n}\left[\varepsilon_e + \frac{\gamma}{n}\varepsilon_{x_0} + \frac{\gamma^2}{2!n^2}\varepsilon_{x_0^2} + \cdots\right](\{e\}) \\ &\geq \varepsilon_e(\{e\})e^{-\gamma/n} \stackrel{n\uparrow\infty}{\longrightarrow} 1\end{aligned}$$

and

$$\begin{aligned}\mu_n(G\backslash\{e, x_0\}) &= 1 - e^{-\gamma/n}\left[\varepsilon_e + \frac{\gamma}{n}\varepsilon_{x_0} + \frac{\gamma^2}{2!n^2}\varepsilon_{x_0^2} + \cdots\right](\{e, x_0\}) \\ &\leq 1 - e^{-\gamma/n} - \frac{\gamma}{n}e^{-\gamma/n} \stackrel{n\uparrow\infty}{\longrightarrow} 0\,,\end{aligned}$$

(a) and (b) of statement (ii) also hold.

"(ii) $\Rightarrow$ (i)" Let $\mu \in \mathfrak{M}^1(G)$, $\mu \neq \varepsilon_e$, with a sequence $(\mu_n)_{n\geq 1}$ of roots satisfying statement (ii). For proof we will divide into a series of steps.

<u>Step 1</u> : By G_0 we denote the subgroup of G generated by x_0, and we put $\alpha_n = n\mu_n(G\backslash\{e, x_0\})$. Clearly we obtain :

$$\lim_{n\to\infty} \alpha_n = 0\,, \tag{1}$$

$$\mu_n(G_0) \geq \mu_n(\{e, x_0\}) = 1 - \frac{\alpha_n}{n}\,, \tag{2}$$

and

$$\mu(G_0) = \mu_n^n(G_0) \geq \left(1 - \frac{\alpha_n}{n}\right)^n \quad \text{for all } n \geq 1. \tag{3}$$

Since there exists an n_0 with $\frac{\alpha_n}{n} < 1$ for all $n \geq n_0$, we have :

$$1 - \alpha_n \leq \left(1 - \frac{\alpha_n}{n}\right)^n \leq 1 \qquad \text{for all } n \geq n_0\,,$$

and hence by (1) $\lim_{n\to\infty}\left(1-\frac{\alpha_n}{n}\right)^n = 1$. That is, $\mu(G_0) = 1$. Since G_0 is at most countable, μ_n is a countable measure for all $n \geq 1$. Therefore without loss generality we assume that G is discrete.

Step 2 : There exists an $n_0 \geq 1$ such that $\mu_n(\{e\}) > 0$ for all $n \geq n_0$. If it were not, then there would exist a subsequence $(n_k)_{k\geq 1}$ such that $\mu_{n_k}(\{e\}) = 0$ for all $k \geq 1$. This implies $\mu_{n_k}(\{x_0\}) = 1 - \frac{\alpha_{n_k}}{n_k}$. Now without loss generality we assume $\left(1-\frac{\alpha_{n_k}}{n_k}\right)^{n_k} \geq \frac{3}{4}$ for all $k \geq 1$. Then we have :

$$\begin{aligned}\mu(\{x_0^{n_k}\}) &= \mu_{n_k}^{n_k}(\{x_0\}^{n_k}) \geq \mu_{n_k}^{n_k}(\{x_0\}) \\ &= \left(1-\frac{\alpha_{n_k}}{n_k}\right)^{n_k} \geq \frac{3}{4}\end{aligned}$$

for all $k \geq 1$. This yields $x_0^{n_k} = x_0^{n_1}$ for all k, and hence

$$\mu(\{x_0^{n_1}\}) = \lim_{k\to\infty}\mu(\{x_0^{n_k}\}) \geq \lim_{k\to\infty}\left(1-\frac{\alpha_{n_k}}{n_k}\right)^{n_k} = 1,$$

i.e., $\mu = \varepsilon_{y_1}$ where $y_1 = x_0^{n_1}$. From this we obtain $\mu_n = \varepsilon_{y_n}$, $y_n \in G$ for all $n \geq 1$. Especially by our assumption $\frac{\alpha_{n_k}}{n_k} < 1$, and so $\mu_{n_k}(\{x_0\}) > 0$ for all $k \geq 1$. Hence we have $y_{n_k} = x_0$ for every k. On the other hand if there exists an $m \geq 1$ with $\mu_m(\{e\}) > 0$, then $\mu_m = \varepsilon_e$ and thus $\mu = \varepsilon_e$. Thus if we choose $n_k = m_0 + k$ with a sufficiently large m_0 for all $k \geq 1$, then we obtain :

$$x_0^{m_0+1} = x_0^{n_1} = x_0^{n_2} = x_0^{m_0+2}$$

i.e., $x_0 = e$. But this contradicts $x_0^2 \neq e$, so $\mu_n(\{e\}) = 0$ for all n. However by (a) of statement (ii) this is impossible, that is, there exists an $n_0 \geq 1$ such that $\mu_n(\{e\}) > 0$ for all $n \geq n_0$.

Step 3 : For the above n_0, $e \in \text{supp}(\mu_n)$ for $n \geq n_0$ and hence

$$\text{supp}(\mu_n) \subset [\text{supp}(\mu_n)]^n = \text{supp}\left(\mu_n^n\right) = \text{supp}(\mu) \subset G_0 .$$

Furthermore we have $\left(\mu_{nn_0}^{n_0}\right)^n = \mu$ and $\text{supp}\left(\mu_{nn_0}^{n_0}\right) \subset G_0$ since $nn_0 \geq n_0$. Hence we can replace μ_n by μ_{nn_0} for $1 < n < n_0$. In this case clearly condition (b) is preserved. So we can assume, without loss generality, that G is the discrete group generated by x_0. Thus G is cyclic and Abelian.

Step 4 : Let $\tilde{\mu_n}$ be a adjoint measure for μ_n and $\lambda_n = \mu_n * \tilde{\mu}_n$ for all $n \geq 1$. Then

$$\lambda_n^\wedge(\chi) = |\mu_n^\wedge(\chi)|^2 = \left(|\mu_1^\wedge(\chi)|^2\right)^{1/n}, \qquad n = 1,2,3,\cdots, \chi \in \widehat{G} \quad (4)$$

and for $n = 1, 2, 3, \cdots$,

$$\begin{aligned}\lambda_n(G\backslash\{e, x_0, x_0^{-1}\}) &= \int_{G\backslash\{e,x_0\}} \mu_n(Gx\backslash\{e, x_0, x_0^{-1}\}x)\mu_n(dx) \\ &\quad + \mu_n(G\backslash\{e, x_0, x_0^2\})\mu_n(\{x_0\}) + \mu_n(G\backslash\{e, x_0, x_0^{-1}\})\mu_n(\{e\}) \\ &\leq 3\mu_n(G\backslash\{e, x_0\}) \leq 3n\mu_n(G\backslash\{e, x_0\}).\end{aligned}$$

Hence, by condition (b), we have :

$$\lim_{n\to\infty} \lambda_n(G\backslash\{e, x_0, x_0^{-1}\}) = 0. \tag{5}$$

Moreover by (4) it follows that the limit

$$\lambda^\wedge(\chi) = \lim_{n\to\infty} \lambda_n^\wedge(\chi), \qquad \chi \in \widehat{G} \tag{6}$$

exists and

$$\lambda^{\wedge 2}(\chi) = \lambda^\wedge(\chi) \qquad \text{for all } \chi \in \widehat{G}. \tag{7}$$

Since G is discrete, G is finite or countable. If G is countable, $\lambda_1^\wedge(\chi) \neq 0$ for all $\chi \in \widehat{G}$, by Theorem 4.2([5],p78), since λ_n has no idempotent factor. So we have $\mathcal{T}_w - \lim_{n\to\infty} \lambda_n = \varepsilon_e$ by Theorem 3.3([5],p76). If G is finite, there exists an idempotent factor $\alpha \in \mathfrak{M}^1(G)$ such that $\alpha^\wedge(\chi) = \lambda^\wedge(\chi)$ by Theorem 3.2([5],p75). Hence we have $\mathcal{T}_w - \lim_n \lambda_n = \alpha$ by Theorem 3.3([5],p76). However supp(α) is a subgroup of G by Theorem 3.1([5],p62) and $x_0^2 \neq e$ by condition, we must have $\alpha = \varepsilon_e$. In any case of G we obtain $\mathcal{T}_w - \lim_{n\to\infty} \lambda_n = \varepsilon_e$. Consequently there exists an $n_0 \geq 1$ such that $\|\mu_{n_0} * \tilde{\mu}_{n_0} - \varepsilon_e\| < 1$. Hence conditions of the Theorem 6.1.22 in [2] are completely held, and so μ is a Poisson measure in $\mathcal{P}_{\{e\}}(G)$ with parameter x_0. ∎

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