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THE FILTERS OF THE ORDERED Γ-SEMIGROUPS

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ABSTRACT. We give the relation between the semilattice congruence $\mathcal{N}$ and the set of prime ideals of the ordered Γ-semigroup M.

1. Introduction.

In 1986, Sen and Saha([3],[4]) introduced the concept of a Γ-semigroup and obtained some interesting results. In 1987, Kehayopulu[1] introduced the concept of a filter in poe-semigroups. Later Kehayopulu[2] defined the relation $\mathcal{N}$ on a po-semigroup S as follows:

$$a\mathcal{N}b \Longleftrightarrow N(a) = N(b),$$

where $N(a)$ is the filter of S generated by $a (a \in S)$ and obtained the Proposition.

Proposition [2]. *Let S be a po-semigroup. The following statements hold true:*
(1) *$\mathcal{N}$ is a semilattice congruence on S.*
(2) *$\mathcal{N} = \bigcap\{\sigma_I | I \in \mathcal{I}(S)\}$ where $\mathcal{I}(S)$ is the set of prime ideals of S.*

In this paper we proved the same result in a po-Γ-semigroups.

A *po-Γ-semigroup (: ordered Γ-semigroup)*[5] is an ordered set M at the same time a Γ-semigroup such that

$$a \leq b \Longrightarrow a\gamma x \leq b\gamma x \quad \text{and} \quad x\mu a \leq x\mu b$$

for all $a, b, x \in M$ and for all $\gamma, \mu \in \Gamma$.

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II. Main results.

We recall some definitions.

Definition 2.1 [3]. Let $M = \{a, b, c, \cdots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \cdots\}$ be two nonempty sets. M is called a Γ-*semigroup* if

(1) $a\alpha b \in M$ for $\alpha \in \Gamma$ and $a, b \in M$ and

(2) $(a\alpha b)\beta c = a\alpha(b\beta c)$,

for all $a, b, c \in M$ and for all $\alpha, \beta \in \Gamma$.

Example 2.2 [3]. Let A and B be two non-empty sets, M the set of all mappings from A to B, and Γ a set of some mappings from B to A. The usual mapping product of two elements of M cannot be defined. But if we take f, g from M and α from Γ then the usual mapping product $f\alpha g$ can be defined. Also we find that $f\alpha g \in M$ and $(f\alpha g)\beta h = f\alpha(g\beta h)$ for $f, g, h \in M$ and $\alpha, \beta \in \Gamma$.

Example 2.3 [3]. Let M be a set of $m \times n$ matrices and Γ a set of some $n \times m$ matrices. If we define $A_{m,n}\alpha_{n.m}B_{m,n}$ the usual product of matrices, then

$$(A_{m,n}\alpha_{n,m}B_{m,n})\beta_{n,m}C_{m,n} = A_{m,n}\alpha_{n,m}(B_{m,n}\beta_{n,m}C_{m,n})$$

where $A_{m,n}, B_{m,n}, C_{m,n} \in M$ and $\alpha_{n,m}, \beta_{n,m} \in \Gamma$. Therefore M is a Γ-semigroup.

Let M be a po-Γ-semigroup. An equivalence relation $\Re$ on M is called *congruence* if

$$(a, b) \in \Re \Longrightarrow (a\gamma c, b\gamma c) \in \Re, \quad (c\gamma a, c\gamma b) \in \Re$$

for all $\gamma \in \Gamma$ and $c \in M$.

A congruence $\Re$ on M is called *semilattice congruence* if

$$(a\gamma a, a) \in \Re \text{ and } (a\gamma b, b\gamma a) \in \Re$$

for all $\gamma \in \Gamma$ and $a, b \in M$.

Definition 2.4. Let M be a po-Γ-semigroup and F a sub-Γ-semigroup. Then F is called a *filter* of M if

(1) $a, b \in M, a\gamma b \in F(\gamma \in \Gamma) \Longrightarrow a \in F$ and $b \in F$.

(2) $a \in F, a \leq c(c \in M) \Longrightarrow c \in F$.

For $A, B \subseteq M$, let $A\Gamma B := \{a\gamma b | a \in A, b \in B, \gamma \in \Gamma\}$.

Definition 2.5. Let M be a po-Γ-semigroup and A a nonempty subset of M. A is called a *right*(resp.*left) ideal* of M if

(1) $A\Gamma M \subseteq A$(resp. $M\Gamma A \subseteq A$).

(2) $a \in A, b \leq a$ for $b \in M \Longrightarrow b \in A$.

A is called an *ideal* of M if it is a right and left ideal of M.

Definition 2.6. Let M be a po-Γ-semigroup and T an ideal of M. T is called *prime* if $A, B \subseteq M, A\Gamma B \subseteq T \Longrightarrow A \subseteq T$ or $B \subseteq T$.

Equivalent Definition:

$$a, b \in M, a\gamma b \in T(\gamma \in \Gamma) \Longrightarrow a \in T \quad \text{or} \quad b \in T.$$

Throughout this paper, we denote by $M\backslash F$ the complement of F to M.

Lemma. *Let M be a po-Γ-semigroup and let $\emptyset \neq F \subseteq M$. The following are equivalent:*

(1) *F is a filter of M.*

(2) *$M\backslash F = \emptyset$ or $M\backslash F$ is a prime ideal of M.*

Proof. (1) $\Longrightarrow$ (2). Let $M\backslash F \neq \emptyset$. Then $M\backslash F$ is a prime ideal of M. In fact: Since $M\backslash F \neq \emptyset$, we take $a \in M, b \in M\backslash F$. If $a\gamma b \in F$ for some $\gamma \in \Gamma$, then,since F is a filter of M, we have $a \in F$ and $b \in F$. It is impossible. Thus we have $a\gamma a \in M\backslash F$ for all $\gamma \in \Gamma$, i.e. $M\Gamma(M\backslash F) \subseteq M\backslash F$. Similarly we get $(M\backslash F)\Gamma M \subseteq M\backslash F$. Let $a \in M\backslash F$ and $b \leq a, b \in M$. If $b \in F$ then, since $b \leq a$ and $b \in F$, F is a filter of M, we have $a \in F$. It is impossible. Thus we get $b \in M\backslash F$. Moreover, let $a, b \in M$ and $a\gamma b \in M\backslash F(\gamma \in \Gamma)$. If $a \in F$ and $b \in F$ then,since F is a sub-Γ-semigroup of M, $a\gamma b \in F$. It is impossible. Hence we have $a \in M\backslash F$ or $b \in M\backslash F$.

(2) $\Longrightarrow$ (1). Let $M\backslash F = \emptyset$. Since $M = F$, F is a filter of M. Suppose that $M\backslash F$ is a prime ideal of M. Then F is a sub-Γ-semigroup of M. In fact: Let $a, b \in F$. If $a\gamma b \in M\backslash F(\gamma \in \Gamma)$, since $M\backslash F$ is prime, $a \in M\backslash F$ or $b \in M\backslash F$. It is impossible. Thus we have $a\gamma b \in F$ for all $\gamma \in \Gamma$. Let $a, b \in M$ and $a\gamma b \in F$ for any $\gamma \in \Gamma$. If $a \in M\backslash F$ then, since $M\backslash F$ is an ideal of M, $a\gamma b \in M\backslash F$ for all $\gamma \in \Gamma$. It is impossible. If $b \in M\backslash F$, similarly, $a\gamma b \in M\backslash F$ for all $\gamma \in \Gamma$. It is impossible. Thus we have $a \in F$ and $b \in F$. Let $a \in F$ and $a \leq b, b \in M$. If $b \in M\backslash F$ then, since $a \leq b, b \in M\backslash F$ and $M\backslash F$ is an ideal of M, we get $a \in M\backslash F$. It is impossible. Thus we have $b \in F$. Therefore F is a filter of M.

Let M be a po-Γ-semigroup and I a prime ideal of M. We define a relation $\Re_I$ on M as follows:

$$\Re_I = \{(a,b) | a, b \in I \quad \text{or} \quad a, b \notin I\}.$$

Then $\Re_I$ is a semilattice congruence on M. We denote by $N(a)$ the filter of M generated by $a (a \in M)$. We denote by $\mathcal{N}$ the equivalence relation on M defined by :

$$\mathcal{N} = \{(a,b) | N(a) = N(b)\}$$

Theorem 2.7. *Let M be a po-Γ-semigroup . The following statements hold true:*
(1) *$\mathcal{N}$ is a semilattice congruence on M.*
(2) $\mathcal{N} = \bigcap\{\Re_I | I \in \mathcal{I}(M)\}$
where $\mathcal{I}(M)$ is the set of prime ideals of M.

Proof. (1). Let $(a,b) \in \mathcal{N}, c \in M$. For all $\gamma \in \Gamma$, since $c\gamma a \in N(c\gamma a)$, we have $c, a \in N(c\gamma a)$. Since $a \in N(c\gamma a)$, $N(a) \subseteq N(c\gamma a)$ and thus $b \in N(c\gamma a)$ for all $\gamma \in \Gamma$. Since $c, b \in N(c\gamma a)$, we have $c\mu b \in N(c\gamma a)$ for all $\gamma, \mu \in \Gamma$ and thus $N(c\mu b) \subseteq N(c\gamma a)$ for all $\gamma, \mu \in \Gamma$.

By symmetry, $N(c\gamma a) \subseteq N(c\mu b)$ for all $\gamma, \mu \in \Gamma$. Thus $N(c\gamma a) = N(c\gamma b)$, i.e. $(c\gamma a, c\gamma b) \in \mathcal{N}$. Similarly, $(a\gamma c, c\gamma b) \in \mathcal{N}$ for all $c \in M, \gamma \in \Gamma$. Similarly,$(a\gamma c, b\gamma c) \in \mathcal{N}$ for all $c \in M, \gamma \in \Gamma$.

Let $a \in M$. Since $a\gamma a \in N(a\gamma a)$ for all $\gamma \in \Gamma$ we have $a \in N(a\gamma a)$,and $N(a) \subseteq N(a\gamma a)$ for all $\gamma \in \Gamma$. Since $a \in N(a)$, we have $a\gamma a \in N(a)$, and $N(a\gamma a) \subseteq N(a)$. Then $(a\gamma a, a) \in \mathcal{N}$ for all $\gamma \in \Gamma$.

Let $a, b \in M$. Since $a\gamma b \in N(a\gamma b)$ for all $\gamma \in \Gamma$, we have $a, b \in N(a\gamma b)$ and $b\gamma a \in N(a\gamma b)$ for all $\gamma \in \Gamma$. Thus we get $N(b\gamma a) \subseteq N(a\gamma b)$ for all $\gamma \in \Gamma$. By symmetry, $N(a\gamma b) \subseteq N(b\gamma a)$. Thus we have $(a\gamma b, b\gamma a) \in \mathcal{N}$.

(2). Let $(a,b) \in \mathcal{N}$ and $I \in \mathcal{I}(M)$. Let $(a,b) \notin \Re_I$. Then $a \notin I$ and $b \in I$ or $a \in I$ and $b \notin I$. Let $a \notin I$ and $b \in I$. Then $\emptyset \neq M\backslash I \subseteq M$ and $a \in M\backslash I$. Since $M\backslash(M\backslash I) = I$, $M\backslash(M\backslash I)$ is a prime ideal of M. By the Lemma, $M\backslash I$ is a filter of M. Since $a \in M\backslash I$, we have $N(a) \subseteq M\backslash I$ and thus $b \in M\backslash I$. It is impossible. Similarly, from $a \notin I$ and $b \in I$, we get a contradiction. Thus we have $\mathcal{N} \subseteq \bigcap_{I \in \mathcal{I}(M)} \Re_I$.

Conversely, let $(a,b) \in \Re_I$ for all $I \in \mathcal{I}(M)$. If $(a,b) \notin \mathcal{N}$, then $a \notin N(b)$ or $b \notin N(a)$. In fact, if $a \in N(b)$ and $b \in N(a)$, then $N(a) \subseteq N(b)$ and $N(b) \subseteq N(a)$ and so $(a,b) \notin \mathcal{N}$. Let $a \notin N(b)$. Then $a \in M\backslash N(b)$ and thus $M\backslash N(b) \neq \emptyset$. Since

$N(b)$ is a filter of M, by the Lemma, $M\backslash N(b)$ is a prime ideal of M. Therefore we have $M\backslash N(b) \in \mathcal{I}(M), a \in M\backslash N(b)$ and $b \notin M\backslash N(b)$, i.e. $M\backslash N(b) \in \mathcal{I}(M)$ and $(a,b) \notin \Re_{M\backslash N(b)}$. We get a contradiction. Similarly, from $b \notin N(a)$, we have a contradiction.

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