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SOME GENERALIZATION OF MINTY'S LEMMA

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ABSTRACT. We obtain a generalization of Behera and Panda's result on nonlinear scalar case to the vector version.

1. Introduction and preliminaries

In the recent decades, there have been a great deal of developments in the theory of optimization techniques. The study of variational inequalities and complementarity problems is also a part of this development because optimization problems can often be reduced to the solution of variational inequalities and complementarity problems.

One of the important results of variational inequality theory is Minty's Lemma, which has interesting applications in the study of obstacles problems, confined plasmas, filtration phenomena, free-boundary problems, plasticity and viscoplasticity phenomena, elasticity problems, stochastic optimal control problems and others.

Let $\langle \cdot,\cdot \rangle : X^* \times X \to \mathbb{R}$ be the duality pairing.

The classical Minty's Lemma (cf. [2] and [3]) is stated as follows:

Theorem 1.1. *Let X be a reflexive real Banach space, K a nonempty closed convex subset of X and X^* be the dual of X. Let $T: K \to X^*$ be a monotone operator which is continuous on finite dimensional subspaces (or at least hemicontinuous). Then the followings are equivalent.*

(a) *There exists an $x_0 \in K$ such that*

$$\langle T(x_0), y - x_0 \rangle \geq 0 \quad \textit{for all} \quad y \in K.$$

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(b) *There exists an* $x_0 \in K$ *such that*

$$\langle T(y), y - x_0 \rangle \geq 0 \quad \textit{for all} \quad y \in K.$$

Recently, Theorem 1.1 was generalized by Behera and Panda [1] to the nonlinear case.

In this paper, we obtain a generalization of Behera and Panda's result on nonlinear scalar case to the vector version.

Let X and Y be two topological vector spaces, K a nonempty convex subset of X, and C be a set-valued mapping from K into 2^Y such that for every $x \in K$, $C(x)$ is a closed convex solid cone of Y, i.e., its interior is nonempty. Let T be an operator of X into the space $L(X, Y)$ of all continuous linear operators, and $\eta : K \times K \to X$ be an operator. We define an order relation $\leq_{C(x)}$ in Y by the convex cone $C(x)$ as follows: for $y_1, y_2 \in Y$, $y_1 \leq_{C(x)} y_2 \Leftrightarrow y_2 - y_1 \in C(x)$.

Definition 1.1. Let $T: K \to L(X, Y)$ be an operator. T is said to be *η-hemicontinuous* in the sense of Stampacchia [5] if, for any x, $y \in K$, $\alpha \in [0, 1]$, the mapping

$$\alpha \longmapsto \langle T(\alpha y + (1 - \alpha)x), \eta(y, x) \rangle$$

is continuous, where $\langle \cdot, \cdot \rangle: L(X, Y) \times X \to Y$ is the duality pairing.

Definition 1.2 [3]. An operator $G: K \to Y$ is said to be *continuous on finite dimensional subspaces* if for every finite dimensional subspace M of X, the operator $G: K \cap M \to Y$ is weakly continuous.

Definition 1.3 [4]. An operator $G: X \to Y$ is said to be *$C(x)$-convex* if for every $x_1, x_2 \in X$ and $\lambda \in (0, 1)$,

$$G(\lambda x_1 + (1 - \lambda)x_2) \leq_{C(x)} \lambda G(x_1) + (1 - \lambda)G(x_2),$$

i.e., $\lambda G(x_1) + (1 - \lambda)G(x_2) - G(\lambda x_1 + (1 - \lambda)x_2) \in C(x)$.

2. Main Results

Now we consider the following generalized vector version of Minty's Lemma.

Theorem 2.1. *Let X and Y be topological vector spaces, K a nonempty convex subset of X, and $\{C(x): x \in K\}$ a family of closed convex solid cones of Y.*

Suppose that $T: K \to L(X,Y)$ and $\eta: K \times K \to X$ be operators such that

(i) $\langle T(y), \eta(y,y)\rangle \notin -\operatorname{Int} C(x)$ *for all $x, y \in K$,*

(ii) *the operator*

$$x \longmapsto \langle T(x), \eta(y,x)\rangle$$

of K into Y is continuous on finite dimensional subspaces (or at least η-hemicontinuous) for each $y \in K$,

(iii) *the operator*

$$y \longmapsto \langle T(x), \eta(y,x)\rangle$$

of K into Y is convex for each $x \in K$,

(iv) $\langle T(x), \eta(y,x)\rangle + \langle T(y), \eta(x,y)\rangle \notin \operatorname{Int} C(x)$ *for all x, $y \in K$.*

Then the followings are equivalent:

(a) *there exists an $x_0 \in K$ such that*

$$\langle T(x_0), \eta(y,x_0)\rangle \notin -\operatorname{Int} C(x_0)$$

for all $y \in K$.

(b) *there exists an $x_0 \in K$ such that*

$$\langle T(y), \eta(x_0,y)\rangle \notin \operatorname{Int} C(x_0)$$

for all $y \in K$.

Proof. Suppose that there exists an $x_0 \in K$ satisfying

$$\langle T(x_0), \eta(y,x_0)\rangle \notin -\operatorname{Int} C(x_0)$$

for all $y \in K$. Then it is easily shown that for such an x_0 we have $\langle T(y), \eta(x_0,y)\rangle \notin \operatorname{Int} C(x_0)$ for all $y \in K$ from (iv) by the fact that $\operatorname{Int} C(x) + \operatorname{Int} C(x) = \operatorname{Int} C(x)$.

Conversely, suppose that there exists an $x_0 \in K$ such that

$$\langle T(y), \eta(x_0,y)\rangle \notin \operatorname{Int} C(x_0)$$

for all $y \in K$. For arbitrary $x \in K$, letting $y_\lambda = \lambda x + (1-\lambda)x_0$, $0 < \lambda < 1$, we have a $y_\lambda \in K$ by the convexity of K. Hence

$$\langle T(y_\lambda), \eta(x_0,y_\lambda)\rangle \notin \operatorname{Int} C(x_0).$$

On the other hand, by the convexity of the operator

$$y \longmapsto \langle T(x), \eta(y,x) \rangle,$$

we have

$$\lambda\langle T(y_\lambda), \eta(x, y_\lambda)\rangle + (1-\lambda)\langle T(y_\lambda), \eta(x_0, y_\lambda)\rangle - \langle T(y_\lambda), \eta(y_\lambda, y_\lambda)\rangle \\ \notin -\operatorname{Int} C(x_0).$$

Since

$$\langle T(y_\lambda), \eta(x_0, y_\lambda)\rangle \notin \operatorname{Int} C(x_0),$$

we have

$$\langle T(y_\lambda), \eta(x, y_\lambda)\rangle \notin -\operatorname{Int} C(x_0) \tag{2.1}$$

from (i).
Since the operator

$$x \longmapsto \langle T(x), \eta(y,x) \rangle$$

of K into Y is continuous on finite dimensional subspaces (or at least η-hemicontinuous), taking the limit as λ approaches to 0^+ in (2.1) we get

$$\langle T(x_0), \eta(y, x_0)\rangle \notin -\operatorname{Int} C(x_0)$$

for all $y \in K$ by the closedness of $Y \backslash (-\operatorname{Int} C(x_0))$. □

Corollary 2.2 [1]. *Let X be a nonempty closed convex subset of a reflexive real Banach space X and let X^* be the dual of X. Let $T : K \to X^*$ and $\theta : K \times K \to X$ be two maps such that*

(i) $\langle T(y), \theta(y,y)\rangle = 0$ *for all* $y \in K$,

(ii) *the map*

$$x \longmapsto \langle T(x), \theta(y,x)\rangle$$

of K into $\mathbb{R}$ is continuous on finite dimensional subspaces (or at least hemicontinuous), for each $y \in K$,

(iii) *the map*

$$y \longmapsto \langle T(x), \theta(y,x)\rangle$$

of K into $\mathbb{R}$ is convex for each $x \in K$,

(iv) $\langle T(x), \theta(y,x)\rangle + \langle T(y), \theta(x,y)\rangle \leq 0$ *for all* $x, y \in K$.

Then the followings are equivalent:

(a) $x_0 \in K, \langle T(x_0), \theta(y, x_0) \rangle \geq 0$ *for all* $y \in K$.

(b) $x_0 \in K, \langle T(y), \theta(x_0, y) \rangle \leq 0$ *for all* $y \in K$.

Proof. If we put $C(x) = [0, \infty)$ in Theorem 2.1, it can be easily shown. □

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