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SOME PROPERTIES OF A DIRECT INJECTIVE MODULE

CHANG WOO HAN AND SU JEONG CHOI

ABSTRACT. The purpose of this paper is to show that by the divisibility of a direct injective module, we obtain some results with respect to a direct injective module.

1. Introduction

Throughout this paper, let R be a ring. All modules are unitary left R-modules and all maps are R-homomorphisms. A module M is said to be *direct injective* if, given any direct summand N of M with an inclusion $i : N \to M$, for each monomorphism $f : N \to M$, there exists an endomorphism g of M such that the following diagram

$$\begin{array}{ccccc} & & M & & \\ & & i\uparrow & \nwarrow g & \\ O & \longrightarrow & N & \xrightarrow{f} & M \end{array}$$

commutes, i.e., $g \circ f = i$. The concept of a direct injective module as the generalization of a quasi-injective module was introduced by Nicholson [3] in 1976.

Xue [5] showed the characterizations of hereditary ring and semisimple ring by using direct projective modules and direct injective modules. A module M is said to have the *summand sum property* if the sum of any two direct summands of M is again a direct summand of M. Similarly, a module M is said to have the *summand*

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intersection property if the intersection of any two direct summands of M is a direct summand of M.

In this paper, we show that every direct summand of a direct injective module is direct injective. Through the divisibility of a direct injective module, we have some properties of a direct injective module. In addition, we prove that every module which has the summand intersection property has the summand sum property.

2. Results

Theorem 2.1. *Every direct summand of a direct injective module is direct injective.*

Proof. Assume that M is a direct injective module. Let N be a direct summand of M. Given any direct summand A of N, monomorphisms $f : A \to N$ and $g : N \to M$, and the inclusion maps $i_A : A \to N$ and $i_N : N \to M$,

$$\begin{array}{ccccccc} & & M & & & & \\ & & \uparrow{\scriptstyle i_N} & & & & \\ & & N & & & & \\ & & \uparrow{\scriptstyle i_A} & & & & \\ O & \longrightarrow & A & \xrightarrow{f} & N & \xrightarrow{g} & M \end{array}$$

since M is a direct injective module, there exists an endomorphism k of M such that $k \circ g \circ f = i_N \circ i_A$. We define an endomorphism h of N by $h = p_N \circ k \circ g$ and so we obtain the following diagram

$$\begin{array}{ccccc} & & N & & \\ & & \uparrow{\scriptstyle i_A} & & \\ O & \longrightarrow & A & \xrightarrow{f} & N \end{array}$$

commutes, i.e.,

$$h \circ f = (p_N \circ k \circ g) \circ f = p_N \circ (k \circ g \circ f) = (p_N \circ i_N) \circ i_A = I_N \circ i_A = i_A.$$

Therefore, the direct summand N is a direct injective module. □

Theorem 2.2 [2]. *Every dierct injective module M is divisible.*

Corollary 2.3. *If M is a direct injective module, then* $\mathrm{Hom}_Z(R, M)$ *is an injective R-module.*

Proof. Let M be a direct injective module. Then by theorem 2.2, M is a divisible module. If we regard M as an divisible abelian group, then $\mathrm{Hom}_Z(R, M)$ is an injective R-module. □

Corollary 2.4. *Let R be a principal ideal domain. Then M is a direct injective module if and only if M is a divisible module.*

Proof. Assume that M is a direct injective module, then by Theorem 2.2, M is a divisible module.

Conversely, let M be a divisible module. Since R is a principal ideal domain, M is an injective module. This implies that M is a direct injective module. □

Corollary 2.5. *Let R be a principal ideal domain. Then M is an injective module if and only if M is a direct injective module.*

Proof. suppose that M is an injective module. Then clearly M is a direct injective module.

Conversely, let M be a direct injective module. By Theorem 2.2, M is a divisible module. Since R is a principal ideal domain, by Corollary 2.4, M is an injective module. □

The following is related to arbitrary modules.

Theorem 2.6. *For a module M, if M has the summand intersection property, then M has the summand sum property.*

Proof. Assume that a module M has the summand intersection property. It is sufficient to show that for every pair A, B of direct summands of M and the canonical projection $p : M \to B$, $\mathrm{Im}\, p|_A$ is a direct summand of B. Then $\mathrm{Ker}\, p|_A = \mathrm{Ker}\, p \cap A$ is a direct summand of M and by [4, p. 33], $\mathrm{Ker}\, p|_A$ is a direct summand of A. Hence an exact sequence

$$0 \longrightarrow \mathrm{Ker}\, p|_A \longrightarrow A \longrightarrow \mathrm{Im}\, p|_A \longrightarrow 0$$

splits. $\operatorname{Im} p|_A$ is a summand of A and a direct summand of M. $\operatorname{Im} p|_A \subset B \subset M$ implies that $\operatorname{Im} p|_A$ is a direct summand of B. Therefore, M has the summand sum property. □

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DEPARTMENT OF MATHEMATICS, DONG-A UNIVERSITY, HADAN-2-DONG, SAHA-GU, PUSAN 604-714, KOREA.