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PETTIS INTEGRABILITY OF SEPARABLE-LIKE FUNCTIONS

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ABSTRACT. In this paper, we introduce the notion of separable-like function, investigate some properties of separable-like functions, and characterize the Pettis integrability of function on a finite perfect measure space.

1. Introduction

The theory of integration of functions with values in a Banach space has long been a fruitful area of study. The Pettis integral of a weakly measurable function has proved remarkably resistant to analysis since its introduction by B. J. Pettis in 1938. The remarkable progress of the Pettis integral has been achieved by many authors, and the Pettis integral has begun to come into its own.

Geitz [5] characterized Pettis integrable functions on a finite perfect measure space answering, by the way, an old question of Pettis' about the role of simple functions in Pettis integration:

Let (Ω, Σ, μ) be a finite perfect measure space. A bounded weakly μ-measurable function $f : \Omega \to X$ is Pettis integrable if and only if there is an uniformly bounded sequence (f_n) of simple functions from Ω into X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e.

In this paper, we introduce the notion of separable-like function, and investigate some properties of separable-like functions and topics in some more detail. In particular, we study some characterizations of Pettis integrability of function on a finite perfect measure space.

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2. Preliminary definitions and propositions

Let (Ω, Σ, μ) be a finite measure space and let X be a Banach space whose dual space is x^* and bidual is X^{**}.

Let $f : \Omega \to X$ be a bounded weakly μ-measurable function. By the closed graph theorem, $T : X^* \to L^1(\mu)$ defined by $T(x^*) = x^*f$ is a bounded linear operator. Hence, if T^* denotes the adjoint of T, then we can define $T^* : L^\infty(\mu) \to X^{**}$ by

$$T_g^*(x^*) = \int_\Omega gT(x^*)d\mu = \int_\Omega gx^*fd\mu, \quad \text{for each } g \in L^\infty(\mu).$$

In particular, define $F : \Sigma \to X^{**}$ by

$$F(E) = \text{D-}\int_E fd\mu, \quad \text{for each } E \in \Sigma,$$

and $F(E) = T^*\chi_E$ is called the *Dunford integral* of f over E.

The function F is not necessarily countably additive. It can be shown that F is countably additive if and only if T is a weakly compact operator if and only if $\{x^*f : |x^*| \leq 1\}$ is uniformly integrable in $L^1(\mu)$.

In the case that $F(E) \in X$ for each $E \in \Sigma$, we write $F(E) = \text{P-}\int_E fd\mu$ and it is called the *Pettis integral* of f over E. A weakly μ-measurable function $f : \Omega \to X$ is *separable-like* if there exists a separable subspace D of X such that

$$x^*\chi_D f = x^*f \ \mu\text{-a.e.} \quad \text{for every } x^* \in X^*.$$

Huff [6] defined such a separable-like function and investigated the Pettis integrability. By the Pettis Measurability Theorem, every μ-measurable function $f : \Omega \to X$ is separable-like. In particular, simple functions are separable-like.

In [7], we have the following fundamental result.

Theorem 2.1. *Let $f : \Omega \to X$ be Dunford integrable and define $T : X^* \to L^1(\mu)$ by $T(x^*) = x^*f$. Then,*

(a) *T is bounded;*
(b) *if T is weakly compact and f is separable-like, then f is Pettis integrable, and if f is Pettis integrable, then T is weakly compact;*
(c) *T is compact if f is Bochner integrable.*

The condition of the weak compactness of T in the first part of Theorem 2.1(b) is remarkable. If T is not weakly compact, there is an example of a separable-like function which is not Pettis integrable.

Example 2.2. Let $(\Omega, \Sigma, \mu) = ([0, 1]$, Lebesgue measurable sets, Lebesgue measure), and define $f : [0,1] \to X = c_0$ by

$$f(t) = (\chi_{(0,1]}(t), 2\chi_{(0,1/2]}(t), \cdots, n\chi_{(0,1/n]}(t), \cdots), \quad \text{for } t \in [0,1].$$

If $x^* = (\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots)$ is in $c_0^* = l_1$, then

$$x^* f = \sum_{n=1}^{\infty} \alpha_n n \chi_{(0,1/n]}.$$

Hence f is Dunford integrable and separable-like, but it is not Pettis integrable.

In fact, since $\|\text{D-}\int_{(0,1/n]} f d\mu\|_{l_\infty} = 1$ for each n, the map $A \to T^*(\chi_A)$ from Σ to l_∞ is not countably additive, and hence T: $X^* = l_1 \to L^1(\mu)$ is not weakly compact. Hence f is not Pettis integrable.

A finite measure space (Ω, Σ, μ) is *perfect* if for each μ-measurable function Ψ : $\Omega \to \mathbb{R}$ and for each set E in $\mathbb{R}$ such that $\Psi^{-1}(E)$ in Σ, there exists a Borel set B of such that $\mu(\Psi^{-1}(B)) = \mu(\Psi^{-1}(E))$.

The following theorem is a simple translation of [5, Theorem 3, p. 83].

Proposition 2.3. *Let (Ω, Σ, μ) be a finite perfect measure space and let $f : \Omega \to X$ be bounded weakly μ-measurable. Then f is Pettis integrable if there is a sequence (f_n) of Pettis integrable functions from Ω into X such that*

(a) *the set $\{x^* f_n : \|x^*\| \le 1, n \in N\}$ is uniformly integrable, and*

(b) *for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e.*

Joining Diestel and Uhl [2, Theorem 8, p. 55] and Proposition 2.3, we obtain the following proposition.

Proposition 2.4. *Let (Ω, Σ, μ) be a finite perfect measure space and let $f : \Omega \to X$ be Dunford integrable. If there is a sequence (f_n) of Pettis integrable functions from Ω into X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e., then f is Pettis integrable.*

We need the following rather deep result of Fremlin for perfect measure spaces.

Theorem 2.5 (Fremlin). *Let (Ω, Σ, μ) be a finite perfect measure space and (f_n) be a sequence of μ-measurable extended real valued functions on Ω. Then either*

(f_n) has a subsequence which converges μ-a.e. or (f_n) has a subsequence having no μ-measurable pointwise cluster points.

Let (Ω, Σ, μ) be a finite perfect measure space and let $f : \Omega \to X$ be Dunford integrable. If (x_n^*) is any bounded sequence in x^*, then it follows from Alaoglu's Theorem that every subsequence of $(x_n^* f)$ has a pointwise cluster point. Combining this with Fremlin's Theorem, we prove the following lemma.

Lemma 2.6. *If (Ω, Σ, μ) is a finite perfect measure space and $f : \Omega \to X$ is a Dunford integrable function, then the operator $T : X^* \to L^1(\mu)$ defined by $T(x^*) = x^* f$ is compact.*

Proof. Let (x_n^*) be a bounded sequence in X^*. Suppose $(x_n^* f)$ does not have a μ-a.e. convergent subsequence. By Fremlin's theorem, there is a subsequence $(x_{n_j}^* f)$ having no μ-measurable pointwise cluster point. Let x_0^* be a weak* cluster point of $(x_{n_j}^*)$ in X^*. Hence $x_0^* f$ is a pointwise cluster point of $(x_{n_j}^* f)$ and therefore non-measurable. This contradicts the weak measurability of f. Hence some subsequence must converge μ-a.e. and by boundedness this sequence must converge in $L^1(\mu)$, which proves the desired result. □

3. Pettis integrability

Although we remove the weak compactness hypothesis in [7, Theorem 3.9], we get the following theorem:

Theorem 3.1. *Let (Ω, Σ, μ) be a finite perfect measure space and let $f : \Omega \to X$ be Dunford integrable. Then f is Pettis integrable if and only if it is separable-like.*

Proof. Suppose f is Pettis integrable. Then there exists a sequence (f_n) of simple functions from Ω into X such that $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e. by Geitz's theorem (cf. [5, Theorem 6, p. 85]). Note that simple functions are separable-like. For each n, let D_n be a separable subspace of X such that $x^* \chi_{D_n} f_n = x^* f_n$ μ-a.e. for every x^* in X^*. Put $\cup_n D_n = D$. Since $(x^* f_n)$ converges to $x^* f$ μ-a.e., then $x^* \chi_D f = x^* f$ μ-a.e. for every x^* in x^*. Hence f is separable-like. □

For the converse, suppose $f : \Omega \to X$ is Dunford integrable and separable-like. From Lemma 2.6, T is weakly compact. By Theorem 2.1, f is Pettis integrable.

Theorem 3.2. *Let (Ω, Σ, μ) be a finite perfect measure space and let $f : \Omega \to X$ be Dunford integrable. If there exists a sequence of functions (f_n) of Dunford integrable and separable-like from Ω into X such that*

$$x^* f_n = x^* f \quad \mu\text{-a.e.}, \quad \text{for each } x^* \in X^*,$$

then f is Pettis integrable.

Proof. By hypothesis and Theorem 2.1(b), (f_n) is a sequence of Pettis integrable functions from Ω into X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e. Then, by Proposition 2.4, f is Pettis integrable. □

The following theorem offers a characterization of Pettis integrability of functions on a finite perfect measure space.

Theorem 3.3. *Let (Ω, Σ, μ) be a finite perfect measure space and let $f : \Omega \to X$ be Dunford integrable. Then, f is Pettis integrable if and only if there exists a sequence of functions (f_n) of Dunford integrable and separable-like from Ω into X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e.*

Proof. Suppose f is Pettis integrable. Then there exists a sequence (f_n) of simple functions from Ω into X such that, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e. by Geitz's theorem [5, Theorem 6, p. 85]. Note that simple functions are Pettis integrable. Thus there exists a sequence of functions (f_n) of Dunford integrable and separable-like from Ω into X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e. □

The converse is essentially proved in Theorem 3.2. As a corollary of Theorem 3.3, we find the following corollary.

Corollary 3.4. *Let (Ω, Σ, μ) be a finite perfect measure space and $f : \Omega \to X$ be Dunford integrable. Then, f is Pettis integrable if and only if there exists a sequence of functions (f_n) of Pettis integrable from Ω into X such that for each $x^* \in X^*$, $\lim_{n\to\infty} x^* f_n = x^* f$ μ-a.e.*

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