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ON A CLASS OF FUNCTIONS $H_k[A,B]$ DESCRIBED BY SUBORDINATION

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Abstract. We introduce a new class of functions $H_k[A,B]$ described by subordination and we derive a few geometric properties for the class $H_k[A,B]$.

1. Introduction

For $2 \le k \le 4$, let V_k be the class of all functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in $E = \{z : |z| < 1\}$ with $f'(z) \neq 0$ satisfying

$$\limsup_{r\to 1} \int_0^{2\pi} \left| \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \right| d\theta \le k\pi \qquad (z = re^{i\theta},\ 0 < r < 1).$$

V_k is the class of functions with boundary rotation at most $k\pi$. Every function $f \in V_k$ can be given by the Stieltjes integral representation

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + e^{i\theta} z}{1 - e^{i\theta} z} d\psi(\theta),$$

where $\int_0^{2\pi} d\psi(\theta) = 2\pi$ and $\int_0^{2\pi} |d\psi(\theta)| \le k\pi$, $\psi(\theta)$ being a function of bounded variation on $[0, 2\pi]$.

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For $2 \le k \le 4$ let us define the class W_k as follows; $f(z) \in V_k$ if and only if $zf'(z) \in W_k$.

Now, by using a function $g(z)$ in the class V_k of bounded boundary rotation when $2 \le k \le 4$, we shall generalize the concept of strongly close-to-star functions which was introduced originally by Reade [4]. We shall denote this class of functions by H_k. And we shall define a new class $H_k[A,B]$ that is described by subordination.

Definition. Let $f(z)$ be an analytic function in E with $f(0) = 0, f'(0) = 1$. For $-1 \le B < A < 1, f(z)$ belongs to the class $H_k[A,B]$ if there is a function $g(z)$ in V_k such that $\frac{f(z)}{zg'(z)}$ is subordinate to $\frac{1+Az}{1+Bz}$ (in notation, $\frac{f(z)}{zg'(z)} \prec \frac{1+Az}{1+Bz}$).

Equivalently, $f \in H_k[A,B], (-1 \le B < A \le 1)$ if there is a function $h(z) \in W_k$ such that

$$\frac{f(z)}{h(z)} \prec \frac{1+Az}{1+Bz}, \quad z \in E.$$

In this note we reduce a few geometric properties for the new class $H_k[A,B]$.

2. Main results for the class $H_k[A,B]$

For A and B with $-1 \le B < A \le 1$, a function $p(z)$ which is analytic in E with $p(0) = 1$ is said to belong to the class $\mathcal{P}[A,B]$ if $p(z)$ is subordinate to $\frac{1+Az}{1+Bz}$.

Theorem 2.1. *For $f \in H_k[A,B]$ and $|z| \le r < 1$, we have*

$$\frac{1-Ar}{1-Br} \cdot \frac{r(1-r)^{\frac{1}{2}k-1}}{(1+r)^{\frac{1}{2}k+1}} \le |f(z)| \le \frac{1+Ar}{1+Br} \cdot \frac{r(1+r)^{\frac{1}{2}k-1}}{(1-r)^{\frac{1}{2}k+1}}.$$

The bounds are sharp.

Proof. For $f \in H_k[A,B]$, there exists a $h \in W_k$ and $p \in \mathcal{P}[A,B]$ such that

$$f(z) = h(z)p(z). \tag{2.1}$$

Since h is in W_k if and only if $\int_0^z \frac{h(\zeta)}{\zeta} d\zeta$ is in V_k and for $g(z) \in V_k$, $|z| \le r < 1$, we have [3] that

$$\frac{(1-r)^{\frac{1}{2}k-1}}{(1+r)^{\frac{1}{2}k+1}} \le |g'(z)| \le \frac{(1+r)^{\frac{1}{2}k-1}}{(1-r)^{\frac{1}{2}k+1}},$$

and we can write

$$\frac{r(1-r)^{\frac{1}{2}k-1}}{(1+r)^{\frac{1}{2}k+1}} \le |\, h(z)| \le \frac{r(1+r)^{\frac{1}{2}k-1}}{(1-r)^{\frac{1}{2}k+1}}. \tag{2.2}$$

Also for $p \in \mathcal{P}[A,B]$, $|z| \le r$, the univalence of $\frac{1+Az}{1+Bz}$ gives us that

$$\frac{1-Ar}{1-Br} \le |p(z)| \le \frac{1+Ar}{1+Br}. \tag{2.3}$$

Therefore, from (2.1), (2.2) and (2.3) the result follows. Equality is obtained for $f \in H_k[A,B]$ satisfying

$$f(z) = \frac{1+Az}{1+Bz} \cdot \frac{z(1+z)^{\frac{1}{2}k-1}}{(1-z)^{\frac{1}{2}k+1}} \quad \text{and} \quad z = \pm r. \quad \square$$

Theorem 2.2. *For $f \in H_k[A,B]$ and $|z| \le r < 1$,, we have*

$$|\arg f(z)| \le k \cdot \arcsin r + \arcsin \frac{(A-B)r}{1-ABr^2} \qquad (\text{mod } 2\pi).$$

Proof. For $f \in H_k[A,B]$, we have

$$|\arg f(z)| \le |\arg h(z)| + |\arg p(z)|, \tag{2.4}$$

where $h(z) \in W_k$ and $p(z) \in \mathcal{P}[A,B]$. Since $h(z) \in W_k$ *and* $g(z) = \int_0^z \frac{h(\zeta)}{\zeta} d\zeta \in V_k$, we know (cf. [3]) that, for $|z| \le r < 1$,

$$|\arg g'(z)| \le k \cdot \arcsin\ r.$$

Therefore for $h(z) \in W_k$, we have

$$\begin{aligned} |\arg h(z)| &\le |\arg z| + |\arg g'(z)|, \\ &\le k \cdot \arcsin r \qquad (\text{mod } 2\pi). \end{aligned} \tag{2.5}$$

For $p \in \mathcal{P}[A,B]$, $p(|z| < r)$ is contained in the disk

$$\left| p(z) - \frac{1-ABr^2}{1-B^2r^2} \right| < \frac{(A-B)r}{1-B^2r^2} \tag{2.6}$$

from (2.6) it follows that

$$|\arg p(z)| \le \arcsin \frac{(A-B)r}{1-ABr^2}. \tag{2.7}$$

From (2.4), (2.5) and (2.7) we have

$$|\arg f(z)| \le k \cdot \arcsin r + \arcsin \frac{(A-B)r}{1-ABr^2} \qquad (\text{mod } 2\pi). \quad \square$$

Theorem 2.3.

$$H_k[C,D] \subset H_k[A,B]$$

if and only if

$$|AD - BC| \leq (A - B) - (C - D),$$

where $-1 < B < A \leq 1$ *and* $-1 < D < C \leq 1$.

Proof. Since $|z| = 1$ is mapped by $\frac{(1+Az)}{(1+Bz)}$ $(-1 < B < A \leq 1)$ onto a circle centered at $\frac{1-AB}{1-B^2}$ with radius $\frac{A-B}{1-B^2}$, we have

$$H_k[C,D] \subset H_k[A,B]$$

if and only if

$$\{w : \left| w - \frac{1-CD}{1-D^2} \right| < \frac{C-D}{1-D^2}\} \subset \{w : \left| w - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2}\},$$

equivalently it is

$$\frac{1-A}{1-B} \leq \frac{1-C}{1-D} \quad \text{and} \quad \frac{1+C}{1+D} \leq \frac{1+A}{1+B}$$

i.e.,

$$|AD - BC| \leq (A - B) - (C - D). \quad \square$$

The *convolution* of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$.

Theorem 2.4.

$$f \in H_k[A,B]$$

if and only if for all $z \in E$, $h(z) \in W_k, 2 \leq k \leq 4$ *and all* $\zeta, |\zeta| = 1$,

$$\left(f * \frac{(1+B\zeta)z}{1-z}\right) - \left(h * \frac{(1+A\zeta)z}{1-z}\right) \neq 0.$$

Proof. A function f is in $H_k[A,B]$ if and only if there is an $h \in W_k$ such that

$$\frac{f(z)}{h(z)} \neq \frac{1+A\zeta}{1+B\zeta} \quad \text{for } z \in E \text{ and } |\zeta| = 1,$$

which is equivalent to

$$(1+B\zeta)f-(1+A\zeta)h\neq 0$$

or

$$(1+B\zeta)\left(f*\frac{z}{1-z}\right)-(1+A\zeta)\left(h*\frac{z}{1-z}\right)\neq 0.$$

Therefore

$$\left(f*\frac{(1+B\zeta)z}{1-z}\right)-\left(h*\frac{(1+A\zeta)z}{1-z}\right)\neq 0. \quad \square$$

Lemma 1. *If $\phi\in K$ and $g\in V_k$, $2\leq k\leq 4$, then for each function F, analytic in E with $F(0)=1$, the image of E under $\frac{\phi*Fg'}{\phi*g'}$ is a subset of the convex hull of $F(E)$.*

Proof. If g is in V_k, $2\leq k\leq 4$ then we have defined that $zg'(z)$ is in W_k. Due to Ruscheweyh and Sheil-Small [6] it is easy to see that

$$\frac{\phi*Fg'}{\phi*g'}=\frac{z(\phi*Fg')}{z(\phi*g')}=\frac{\phi*F(zg')}{\phi*(zg')}$$

is a subset of the convex hull of $F(E)$. $\square$

Theorem 2.5. *If $f\in H_k[A,B]$, then so is $f*\phi$ for any function $\phi(z)=z+\cdots$, analytic and convex in E.*

Proof. We have $f\in H_k[A,B]$ if and only if, for $z\in E$, there is a function $h\in W_k$ such that

$$\frac{f}{h}\prec\frac{1+Az}{1+Bz}=H,\quad -1\leq B<A\leq 1$$

and equivalently for $z\in E$, there is a function $g\in V_k$ such that

$$F=\frac{f}{zg'}\prec\frac{1+Az}{1+Bz}=H,\quad -1\leq B<A\leq 1.$$

Since H is convex, Lemma 1 yields

$$\frac{(\phi*f)}{(\phi*zg')}=\frac{\phi*Fg'}{\phi*g'}\prec H,$$

so that $\phi*f\in H_k[A,B]$. $\square$

Corollary 1. *For $f \in H_k[A,B]$ and $2 \le k \le 4$, let*

$$F_1(z) = \int_0^z \frac{f(t)}{t}dt, \ F_2(z) = \frac{2}{z}\int_0^z f(t)dt$$

then $F_1(z)$ and $F_2(z)$ are also in $H_k[A,B]$.

Proof. If $F_1(z) = f * \phi_1, F_2(z) = f * \phi_2$, where $\phi_1(z) = -\log(1-z)$ and

$$\phi_2(z) = \frac{-2[z + \log(1-z)]}{z},$$

then trivially ϕ_1 and ϕ_2 are convex and from Theorem 2.5, $F_1(z)$ and $F_2(z)$ are also in $H_k[A,B]$. □

Corollary 2. *If $f \in H_k[A,B]$, $2 \le k \le 4$, then*

$$\frac{1+\gamma}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt, \quad \text{Re}\,\gamma > 0$$

is also in $H_k[A,B]$.

Proof. We can write to use convolution of two power series

$$\frac{1+\gamma}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt = f * \sum_{n=1}^{\infty}\frac{1+\gamma}{n+\gamma}z^n.$$

Since $\sum_{n=1}^{\infty}\frac{1+\gamma}{n+\gamma}z^n$ was shown to be convex by Ruscheweyh and Sheil-Small [6], the result follows from Theorem 2.5. □

Corollary 3. *If $f \in H_k[A,B]$, $2 \le k \le 4$, then*

$$\int_0^z \frac{f(\zeta) - f(x\zeta)}{\zeta - x\zeta}d\zeta \quad (|x| \le 1,\, x \ne 1)$$

is also in $H_k[A,B]$.

Proof. We may write

$$\int_0^z \frac{f(\zeta) - f(x\zeta)}{\zeta - x\zeta}d\zeta = (f * h)(z),$$

where

$$h(z) = \sum_{n=1}^{\infty}\frac{1-x^n}{(1-x)n}z^n = \frac{1}{1-x}\log\frac{1-xz}{1-z} \quad (|x| \le 1,\, x \ne 1).$$

Since h is clearly convex, if $f \in H_k[A,B]$, $2 \le k \le 4$, from Theorem 2.5, so is

$$\int_0^z \frac{f(\zeta) - f(x\zeta)}{\zeta - x\zeta}d\zeta \quad (|x| \le 1,\, x \ne 1). \quad \square$$

Lemma 2 [1]. *Let $f(z) = z + b_2z^2 + \cdots$ be in V_k. Then $|b_2| \le \frac{k}{2}$ and*

$$\max_{f\in V_k} |b_3 - b_2^2| \le \frac{1}{12}\left[\frac{k^2-4}{2} + (k+2) + \frac{k^2-4}{4}\right]$$
$$= \frac{1}{12}\left[\frac{3}{4}(k^2-4) + (k+2)\right]$$

if $2 \le k \le 4$.

Theorem 2.6. *For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $H_k[A,B]$, we have*

$$|a_2| \le k + (A-B) \quad \textit{and} \quad |a_3| \le \frac{(5k-2)(3k+2)}{16} + (A-B)(k+1)$$

if $2 \le k \le 4$.

Proof. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$in $H_k[A,B]$, there exists a function

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in V_k$$

and a Schwarz function $w(z) = \sum_{n=1}^{\infty} \gamma_n z^n$ such that

$$\frac{f(z)}{zg'(z)} = \frac{1+Aw(z)}{1+Bw(z)}, \qquad z \in E.$$

Comparing series expansions,

$$\begin{aligned} a_2 &= 2b_2 + (A-B)\gamma_1, \\ a_3 &= 3b_3 + 2a_2 - 4b_2 + (A-B)\gamma_2 - (A-B)B\gamma_1^2 \\ &= 3b_3 + 2(A-B)b_2\gamma_1 + (A-B)(\gamma_2 - B\gamma_1^2). \end{aligned} \tag{2.8}$$

From the above Lemma 2,

$$|a_2| = |2b_2 + (A-B)\gamma_1| \le |2b_2| + |A-B||\gamma_1| \le k + (A-B).$$

Also by Lemma 2 and (2.8), we have the bound

$$\begin{aligned} |a_3| &\le 3|b_2|^2 + \frac{1}{4}\left|\frac{(3k-2)(k+2)}{4}\right| + 2(A-B)|b_2| + (A-B)\max(1,|B|) \\ &\le \frac{3}{4}k^2 + \frac{3k^2+4k-4}{16} + 2(A-B)\cdot\frac{k}{2} + (A-B) \\ &= \frac{(5k-2)(3k+2)}{16} + (A-B)(k+1), \end{aligned}$$

if $2 \le k \le 4$. □

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