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SOME PHASE PORTRAITS OF PLANAR CONTROL SYSTEMS

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ABSTRACT. In this paper we study periodic orbit of some planar control systems and investigate phase portraits of the FSs

1. Introduction

Consider systems of the form

$$\mathbf{x}' = A\mathbf{x} + \varphi_1(\mathbf{k}\cdot\mathbf{x})\mathbf{b}_1 + \varphi_2(\mathbf{k}\cdot\mathbf{x})\mathbf{b}_2 \tag{1.1}$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is an $\mathbb{R}^2$-valued function, A is a 2×2 matrix and $\mathbf{k} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$, $\mathbf{b}_i = \begin{pmatrix} b_{i1} \\ b_{i2} \end{pmatrix}$ are in $\mathbb{R}^2$ and $\cdot$ denotes the usual inner product.

In this paper, we will focus our attention to studying two-dimensional systems (1.1) satisfying the following conditions:

(a) The origin is an asymptotically equilibrium point;

(b) Two characteristic functions φ_1 and φ_2 have the following forms:

$$\varphi_1(v) = \begin{cases} -u_1 & \text{if} v \le -u_1 \\ v & \text{if} -u_1 \le v \le u_1, \\ u_1 & \text{if} u_1 \le v \end{cases} \quad \varphi_2(v) = \begin{cases} -u_2 & \text{if} v \le -u_2 \\ v & \text{if} -u_2 \le v \le u_2 \\ u_2 & \text{if} u_2 \le v, \end{cases}$$

where u_1, u_2 are fixed positive numbers with $u_1 \le u_2$;

(c) For nonsingular matrix A, $\mathbf{k}\cdot A^{-1}\mathbf{b}_1$ is in $\mathbb{R}$, but for singular matrix A, $\mathbf{b}_1$ and $\mathbf{b}_2$ lie in the same quadrant; and

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(d) For $i = 1, 2$, the followings are hold:

$$t_i = \mathrm{Tr}\,(A + \mathbf{b}_i\mathbf{k}^t) < 0, \qquad d_i = \mathrm{Det}\,(A + \mathbf{b}_i\mathbf{k}^t) > 0.$$

where $\mathbf{k}^t$ is the transpose of $\mathbf{k}$.

Such systems will be referred to as fundamental systems, say FSs, and if $\mathbf{b}_2 = \mathbf{0}$, then our systems (1.1) will correspond to the systems in [3].

The characteristic functions φ_1, φ_2 induce a partition of $\mathbb{R}^2$ into six open strips and four straight lines as follows:

$$S_{1-} := \{\mathbf{x} | \, \mathbf{k}\cdot\mathbf{x} < -u_1\}, \;\; S_{10} := \{\mathbf{x} | -u_1 < \mathbf{k}\cdot\mathbf{x} < u_1\}, \;\; S_{1+} := \{\mathbf{x} | \, \mathbf{k}\cdot\mathbf{x} > u_1\};$$
$$\Gamma_{1-} := \{\mathbf{x} | \, \mathbf{k}\cdot\mathbf{x} = -u_1\}, \; \Gamma_{1+} := \{\mathbf{x} | \, \mathbf{k}\cdot\mathbf{x} = u_1\};$$
$$S_{2-} := \{\mathbf{x} | \, \mathbf{k}\cdot\mathbf{x} < -u_2\}, \;\; S_{20} := \{\mathbf{x} | -u_2 < \mathbf{k}\cdot\mathbf{x} < u_2\}, \;\; S_{2+} := \{\mathbf{x} | \, \mathbf{k}\cdot\mathbf{x} > u_2\};$$
$$\Gamma_{2-} := \{\mathbf{x} | \, \mathbf{k}\cdot\mathbf{x} = -u_2\}, \; \Gamma_{2+} := \{\mathbf{x} | \, \mathbf{k}\cdot\mathbf{x} = u_2\}$$

from which the FS splits into the following linear systems:

$$\mathbf{x}' = A\mathbf{x} - u_1\mathbf{b}_1 + (\mathbf{k}\cdot\mathbf{x})\mathbf{b}_2 \quad \text{in } (S_{1-} \cup \Gamma_{1-}) \setminus S_{2-}, \tag{1.2}$$
$$\mathbf{x}' = A\mathbf{x} - u_1\mathbf{b}_1 - u_2\mathbf{b}_2 \quad \text{in } (S_{2-} \cup \Gamma_{2-}), \tag{1.3}$$
$$\mathbf{x}' = A\mathbf{x} + (\mathbf{k}\cdot\mathbf{x})\mathbf{b}_1 + (\mathbf{k}\cdot\mathbf{x})\mathbf{b}_2 \quad \text{in } (S_{10} \cup \Gamma_{1+}) \cup \Gamma_{1-}, \tag{1.4}$$
$$\mathbf{x}' = A\mathbf{x} + u_1\mathbf{b}_1 + (\mathbf{k}\cdot\mathbf{x})\mathbf{b}_2 \quad \text{in } (S_{1+} \cup \Gamma_{1+}) \setminus S_{2+}, \tag{1.5}$$
$$\mathbf{x}' = A\mathbf{x} + u_1\mathbf{b}_1 + u_2\mathbf{b}_2 \quad \text{in } (S_{2+} \cup \Gamma_{2+}). \tag{1.6}$$

As in (1.2)–(1.6), we can split into linear systems by the characteristic function φ of the control mechanism, the purpose of which is to improve the asymptotic stability behavior of the equilibrium located at the origin.

The main goal of this paper is to describe all the phase portraits of the fundamental system (1.1) and to distinguish among their different qualitative patterns in terms of their basic parameters:

$$t = \mathrm{Tr}(A), \qquad d = \mathrm{Det}(A); \tag{1.7}$$
$$t_i = \mathrm{Tr}(A + \mathbf{b}_i\mathbf{k}^t) < 0, \qquad d_i = \mathrm{Det}(A + \mathbf{b}_i\mathbf{k}^t) > 0 \quad (i = 1, 2); \tag{1.8}$$
$$T = \mathrm{Tr}(B) < 0, \qquad D = \mathrm{Det}(B) > 0 \tag{1.9}$$

where $B = A + \mathbf{b}_1\mathbf{k}^t + \mathbf{b}_2\mathbf{k}^t$. The inequalities (1.8)–(1.9) reflect conditions (a) and (d) in FSs above.

In §2, we classify equilibria of FSs.

In §3, we study periodic orbits of planar differential systems.

In §4, we investigate phase portraits of FSs.

2. Classification of Equilibria

We introduce some known results including Jordan normal forms from those in Llibre and Sotomayor [3] and classify equilibria of the FSs.

Consider

$$\mathbf{x}' = A\mathbf{x} + \varphi_1(\mathbf{k}\cdot\mathbf{x})\mathbf{b}_1 + \varphi_2(\mathbf{k}\cdot\mathbf{x})\mathbf{b}_2. \tag{2.1}$$

By the linear change of variables $\mathbf{x} = M\mathbf{y}$, A can be reduced to its real Jordan form $J = M^{-1}AM$ which transforms the FS into

$$\mathbf{y}' = J\mathbf{y} + \varphi_1(M^t\mathbf{k}\cdot\mathbf{y})M^{-1}\mathbf{b}_1 + \varphi_2(M^t\mathbf{k}\cdot\mathbf{y})M^{-1}\mathbf{b}_2.$$

Replacing the transformed parameters $M^t\mathbf{k}$ by $\mathbf{k}$, $M^{-1}\mathbf{b}_i$ by $\mathbf{b}_i$ and the transformed variable $\mathbf{y}$ by $\mathbf{x}$, the system can be written in the following form similar to (2.1):

$$\mathbf{x}' = J\mathbf{x} + \varphi_1(\mathbf{k}\cdot\mathbf{x})\mathbf{b}_1 + \varphi_2(\mathbf{k}\cdot\mathbf{x})\mathbf{b}_2,$$

where J is one of the following forms:

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ if $d = \mathrm{Det}(A) = 0$ and $t = \mathrm{Tr}(A) = 0$;

$\begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$ if $d = 0$ and $t \neq 0$;

$\begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}$ (with $\beta > 0$) if $d > 0$ $t = 0$;

$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $\begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{bmatrix}$ or $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ (with $\mathrm{sign}(\lambda_i) = \mathrm{sign}(\alpha) = \mathrm{sign}(t)$ and $\beta > 0$) if $d > 0$, $t \neq 0$; and

$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ (with $\lambda_1 < 0 < \lambda_2$) if $d < 0$.

Proposition 1.

(1) *If* $d \geq 0$, *then the origin is the unique equilibrium of the FSs (1.1).*

(2) *If* $d < 0$, *then there are three equilibria for the FS:*

0 *in* S_{10}, *which is always an hyperbolic attractor;*

$$\left.\begin{array}{ll} e_{2+} = & -u_1A^{-1}\mathbf{b}_1 - u_2A^{-1}\mathbf{b}_2 \ \textit{in}\ S_{2+} \\ e_{2-} = & \ \ u_1A^{-1}\mathbf{b}_1 + u_2A^{-1}\mathbf{b}_2 \ \textit{in}\ S_{2-} \end{array}\right\}, \ \textit{which are both saddle points.}$$

Proof. (1) Assume that $d = \mathrm{Det}(B) \neq 0$ and $d_2 = \mathrm{Det}\ (A + \mathbf{b}_2\mathbf{k}^t) \neq 0$.

From (1.2)–(1.6), if $e_{1-} \in S_{1-} \setminus S_{2+}$, $e_{1+} \in S_{1-} \setminus S_{2+}, e_{2-} \in S_{2-}, e_{2+} \in S_{2+}$, then it is easy to see that they are the unique equilibrium points of the FS on $(S_{1-} \cup \Gamma_{1-}) \setminus S_{2-}, (S_{1+} \cup \Gamma_{1+}) \setminus S_{2+}, S_{2-} \cup \Gamma_{2-},\ S_{2+} \cup \Gamma_{2+}$, where

$$e_{1-} = u_1(A + \mathbf{b}_2\mathbf{k}^t)^{-1}\mathbf{b}_1,\ \ e_{1+} = -u_1(A + \mathbf{b}_2\mathbf{k}^t)^{-1}\mathbf{b}_1.$$

Let $\mathbf{k} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$, $J = (a_{ij})$, $\mathbf{b}_1 = \begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix}$ and $\mathbf{b}_2 = \begin{pmatrix} b_{21} \\ b_{22} \end{pmatrix}$. We have the following equivalences:

I.

$$e_{2-} \in S_{2-}, \quad e_{2+} \in S_{2-}$$

if and only if

$$\begin{cases} \mathbf{k} \cdot (u_1J^{-1}b_1) \ \ + \mathbf{k} \cdot (u_2J^{-1}b_2) \ < -u_2, \\ \mathbf{k} \cdot (-u_1J^{-1}b_1) + \mathbf{k} \cdot (-u_2J^{-1}b_2) > \ u_2 \end{cases}$$

if and only if

$$\frac{1}{d}\left[\frac{u_1}{u_2}\left\{(k_1, k_2)\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix}\right\} + \left\{(k_1, k_2)\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{bmatrix} b_{21} \\ b_{12} \end{bmatrix}\right\}\right] < -1.$$

II.

$$e_{1-} \in S_{1-} \setminus S_{2-}, \quad e_{1+} \in S_{1+} \setminus S_{2+}$$

if and only if

$$\begin{cases} -u_2 < \mathbf{k} \cdot \{u_1(J + \mathbf{b}_2\mathbf{k}^t)^{-1}\mathbf{b}_1\} < -u_1, \\ \ \ u_1 < \mathbf{k} \cdot \{-u_1(J + \mathbf{b}_2\mathbf{k}^t)^{-1}\mathbf{b}_1\} < \ u_2 \end{cases}$$

if and only if

$$-\frac{u_2}{u_1} < \frac{1}{d_2}\{k_1(a_{22}b_{11} - a_{12}b_{12}) + k_2(a_{11}b_{12} - a_{21}b_{11})\} < -1.$$

The case $d > 0$ gives us that $e_{2-} \in S_{2-}$, $e_{2-} \in S_{2+}$ if and only if

$$d + \frac{u_1}{u_2}\{k_1(a_{22}b_{11} - a_{12}b_{12}) + k_2(a_{11}b_{12} - a_{21}b_{11})\}$$
$$+\{k_1(a_{22}b_{21} - a_{12}b_{22}) + k_2(a_{11}b_{22} - a_{21}b_{21})\} < 0.$$

Since $\mathbf{k} \cdot A^{-1}\mathbf{b}_1 \le 0$ and $D = \mathrm{Det}\,(A + \mathbf{b}_1\mathbf{k}^t + \mathbf{b}_2\mathbf{k}^t) > 0$ and $\frac{u_1}{u_2} \le 1$, we have $e_{2-} \notin S_{2-}$ and $e_{2+} \notin S_{2+}$.

Since $d_2 > 0$, we find that

$$e_{1-} \in S_{1-} \setminus S_{2-} \quad \text{and} \quad e_{1+} \in S_{1+} \setminus S_{2+} \tag{2.2}$$

if and only if

$$-\frac{u_2}{u_1}d_2 < k_1(a_{11}b_{11} - a_{12}b_{12}) + k_2(a_{11}b_{12} - a_{21}b_{11})\} < -d_2.$$

From above inequality, we see that (2.2) implies

$$d_2 + k_1(a_{11}b_{11} - a_{12}b_{12}) + k_2(a_{11}b_{12} - a_{21}b_{11})\} < 0, \tag{2.3}$$

in view of $d_2 = d + k_1(a_{22}b_{21} - a_{12}b_{22}) + k_2(a_{11}b_{22} - a_{21}b_{21})$, the left side of (2.3) becomes D.

Thus we have $e_{1-} \notin S_{1-} \setminus S_{2-}$ and $e_{1+} \notin S_{1+} \setminus S_{2+}$.

On the other hand the case $d < 0$ gives us that $e_{2-} \in S_{2-}$ and $e_{2+} \in S_{2+}$ if and only if

$$d + \frac{u_1}{u_2}\{k(a_{22}b_{11} - a_{12}b_{12}) + k_2(a_{11}b_{12} - a_{21}b_{11})\}$$
$$+\{k_1(a_{22}b_{21} - a_{12}b_{22}) + k_2(a_{11}b_{22} - a_{21}b_{11})\} > 0$$

if and only if

$$d_2 + \frac{u_1}{u_2}\{k(a_{22}b_{11} - a_{12}b_{12}) + k_2(a_{11}b_{12} - a_{21}b_{11})\} > 0$$

if and only if

$$d_2 + \{k_1(a_{22}b_{11} - a_{12}b_{12}) + k_2(a_{11}b_{12} - a_{21}b_{11})\} > 0 \text{ (see condition (c) of §1)}$$

if and only if

$$D = \mathrm{Det}(A + \mathbf{b}_1\mathbf{k}^t + \mathbf{b}_2\mathbf{k}^t) > 0.$$

Thus e_{2-} and e_{2+} are equilibrium points.

In view of condition (d) of §1, we also see that $e_{1-} \in S_{1-} \setminus S_{2-}$, $e_{1+} \in S_{1+} \setminus S_{2+}$ if and only if

$$-\frac{u_2}{u_1}d_2 < \{k_1(a_{22}b_{11} - a_{12}b_{12}) + k_2(a_{11}b_{12} - a_{21}b_{11})\} < -d_2,$$

which implies

$$d_2 + k_1(a_{22}b_{11} - a_{12}b_{12}) + k_2(a_{11}b_{12} - a_{21}b_{11}) < 0.$$

However $k \cdot A^{-1}b_1 \leq 0$ and $d < 0$, from which we find that $e_{1-} \notin S_{1-} \setminus S_{2-}$ and $e_{1+} \notin S_{1+} \setminus S_{2+}$.

Therefore 0 is the unique equilibrium point when $d > 0$ and 0, e_{2-} and e_{2+} are the unique equilibrium points when $d < 0$.

(2) If $d = 0$, we have only two possibilities for the Jordan form J of A as follows:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \quad (\lambda \neq 0).$$

It follows easily from condition (c) of §1 that 0 is the unique equilibrium point of the FS. □

3. Periodic Orbits of FS

We will study some properties of periodic orbits of planar differential systems.

Proposition 2. *If a fundamental system has a periodic orbit or a double saddle connection, then its basic parameter* $t = \mathrm{Tr}(A) > 0$.

Proof. By the similar argument used in Proposition 3 in [1], any segment on the lines $(\Gamma_{1+}, \Gamma_{1-}, \Gamma_{2+}, \Gamma_{2-})$ cannot be a part of an orbit of the FS.

Therefore, the integrals

$$\oint_{\partial S} X_1 dx_2 - X_2 dx_1 \quad \text{and} \quad \oint\oint_S (\partial X_1/\partial x_1 + \partial X_2/\partial x_2) dx_1 dx_2$$

are well defined for a FS whose components are X_1 and X_2, ∂S is periodic orbit or double saddle connection and S is the region bounded by ∂S.

Since $X_1 dx_2 - X_2 dx_1 = 0$ along ∂S, it follows from Green's formula that

$$\begin{aligned}0 &= \oint\oint_S (\partial X_1/x_1 + \partial X_2/x_2) dx_1 dx_2 \\ &= \oint\oint_{S\cap(S_{2+}\cup S_{2-})} t\, dx_1 dx_2 + \oint\oint_{S\cap\{(S_{1-}\setminus S_{2-})\cup(S_{1+}\setminus S_{2+})\}} t_2\, dx_1 dx_2 \\ &\quad + \oint\oint_{S\cap S_{10}} T\, dx_1 dx_2,\end{aligned}$$

which is written in the following equivalent form:

$$t\,\text{Area}(S\cap(S_{2+}\cup S_{2-})) + t_2\text{Area}[S\cap\{(S_{1-}\setminus S_{2-})\cup(S_{1+}\setminus S_{2+})\}] \\ + T\,\text{Area}(S\cap S_{10}) = 0,$$

which, in virtue of (1.8) and (1.9), immediately yields $t > 0$. □

Our next result shows that fundamental systems with periodic orbits can be written in the form of Lienard systems (cf. [3]), for which a practical criterion for uniqueness will be established. We can prove easily the following proposition by the similar argument used in the proof of Proposition 4 in [3].

Proposition 3. *Assume that a fundamental system*

$$\mathbf{x}' = J\mathbf{x} + \varphi_1(\mathbf{k}\cdot\mathbf{x})\mathbf{b}_1 + \varphi_2(\mathbf{k}\cdot\mathbf{x})\mathbf{b}_2 \tag{3.1}$$

has a periodic orbit or a double saddle connection (DSC) and let $d,\ t,\ d_2,\ D,\ T$ *be its basic parameters (see (1.7)–(1.8)). Then after a linear change from* $\mathbf{x} = (x_1, x_2)$ *to the variables* (x, y) *and a reversing of the time variable, the FS (3.1) can be written in Lienard form:*

$$\begin{cases} x' = y - F(x) \\ y' = -g(x) \end{cases}$$

where

$$g(x) = dx + \varphi_1(x)(d_1 - d) + \varphi_2(x)(d_2 - d)$$

and

$$F(x) = tx + \varphi_1(x)(t_1 - t) + \varphi_2(x)(t_2 - t).$$

Theorem 4 [3]. *Consider the Lienard system*

$$\begin{cases} x' = y - F(x) \\ y' = -g(x) \end{cases} \tag{3.2}$$

which are defined either in the whole $\mathbb{R}^2$ *or in the open strip*

$$S = \{(x, y) \in \mathbb{R}^2 \mid -x_1 - \epsilon < x < x_1 + \varepsilon\}$$

for some $x_1 > 0$ *and* $\varepsilon > 0$.

I. *Suppose that system (3.2) is defined in* $\mathbb{R}^2$ *and satisfies the following assumptions:*

(i) F *and* g *satisfy a Lipschitz condition on any bounded interval;*

(ii) g *is an odd function such that* $xg(x) > 0$ *if* $x \neq 0$; *and*

(iii) F *is an odd function such that there exists* $x_0 > 0$ *for which* $F(x) < 0$ *if* $0 < x < x_0$, *and* $F'(x) > 0$ *if* $x \geq x_0$.

II. *Suppose that the system (3.2) is defined in* S *and satisfies the following assumptions:*

(i)′ F *and* g *satisfy a Lipschitz condition on the interval* $-x_1 - \varepsilon < x < x_1 + \varepsilon$;

(ii)′ g *is an odd function such that* $xg(x) > 0$ *in* $(0, x_1)$ *and* $g(x_1) = 0$;

(iii)′ F *is an odd function such that there exists* $x_0 > 0$ *with* $0 < x_0 < x_1$ *for which* $F(x) < 0$ *if* $0 < x < x_0$, *and* $F'(x) > 0$ *if* $x \geq x_0$; *and*

(iv)′ *The two equilibria* $e_{2+} = (x_1, F(x_1))$, $e_{2-} = (-x_1, F(-x_1))$ *have index* -1 *and the origin has index* 1.

Then, in both cases **I** *and* **II**, *the system (3.2) has at most one periodic orbit.*

Proof. See [3]. □

Using Proposition 3 and Theorem 4, we can show that any fundamental system has at most one periodic orbit.

Corollary 5. *Every FS has at most one periodic orbit.*

Proof. Let d, t, d_i, t_i, D and T be the basic parameters of a given FS (3.1). By Proposition 3, the system (3.1) can be written as the Lienard system (3.2) as follows:

$$\begin{cases} x' = y - F(x) \\ y' = -g(x) \end{cases}$$

where

$$g(x) = dx + \varphi_1(x)(d_1 - d) + \varphi_2(x)(d_2 - d),$$
$$F(x) = tx + \varphi_1(x)(t_1 - t) + \varphi_2(x)(t_2 - t).$$

Since d_i, $t > 0$ and $t_i < 0$, it follows easily that g and F satisfy assumptions (i)-(iii) when $d \geq 0$, and satisfy assumptions (i)′-(iv)′ when $d < 0$. Therefore, by Theorem 4, the corollary is established. □

From Proposition 2, Corollary 5 and Propositions 7 and 9 in [3], we obtain the following theorem.

Theorem 6.

(a) *If the FS (1.1) has a periodic orbit or a double saddle connection, then* $t > 0$. *Moreover, both situations cannot coexist on the same FS;*

(b) *The FS (1.1) has at most one periodic orbit, which moreover must be unstable.*

4. Phase Portraits for FSs

Consider an FS

$$\mathbf{x}' = J\mathbf{x} + \varphi_1(\mathbf{k} \cdot \mathbf{x})\mathbf{b}_1 + \varphi_2(\mathbf{k} \cdot \mathbf{x})\mathbf{b}_2. \tag{4.1}$$

Let d, t, d_i, t_i, D and T be its basic parameters (See (1.7)–(1.8)). The following lemma is the standard integral representation of solutions of differential systems.

Lemma 7. *If* $\mathbf{x}(s)$ *is a solution of a fundamental system (4.1), then*

$$\mathbf{x}(s) = e^{Js}\mathbf{x}(0) + \int_0^s e^{J(s-r)}\{\varphi_1(\mathbf{k} \cdot \mathbf{x}(r))\mathbf{b}_1 + \varphi_2(\mathbf{k} \cdot \mathbf{x}(r))\mathbf{b}_2\}dr.$$

Proof. It follows immediately by differentiation. □

Proposition 8. *If* $d > 0$ *and* $t < 0$, *then* $\Omega(0) = \mathbb{R}^2$, *where* $\Omega(0)$, *being of considerable interest in control theory, denotes the basin of attraction, or stable manifold, of the origin of a FS.*

Proof. From the definitions of J and φ_i there are positive constants L, K_i and r such that $||e^{JS}|| \leq Le^{-rs}$ and $|\varphi_i(\mathbf{k} \cdot \mathbf{x})\mathbf{b}_i| \leq K_i$. Letting $K = \max\{K_1, K_2\}$, we find that

$$|\mathbf{x}(s)| \le Le^{-rs}\left(|\mathbf{x}(0)| + 2K\int_0^s e^r dr\right)$$
$$= Le^{-rs}\left(|\mathbf{x}(0)| - \frac{2K}{r}\right) + \frac{2KL}{r}.$$

Hence, every solution of the FS has its ω-limit set contained in the ball of center 0 and radius $2LK/r$.

From Propositions 1 and 2, the system has the origin as unique equilibrium and does not have periodic orbits. By Poincare-Bendixson theorem, we get $\Omega(0) = \mathbb{R}^2$. □

Proposition 9. *If $d = 0$ and $t < 0$, then $\Omega(0) = \mathbb{R}^2$.*

Proof. We can assume that the matrix J is in Jordan form:

$$\begin{bmatrix} \lambda_- & 0 \\ 0 & 0 \end{bmatrix} \quad (t = \lambda_- < 0).$$

Let $\mathbf{x}(s) = (x_1(s), x_2(s))$ be a solution of the fundamental system. From Lemma 7, we have that

$$|x_i(s)| \le \left|e^{(\lambda_-)s}\right| |x_1(0)| + \int_0^s \left|e^{(\lambda_-)(s-r)}\right| (|\varphi_1(\mathbf{k}\cdot\mathbf{x}(r)\mathbf{b}_1| + |\varphi_2(\mathbf{k}\cdot\mathbf{x}(r))\mathbf{b}_2|)dr.$$

By the definition of φ_i, there are two positive constants K_1, K_2 such that

$$|\varphi_1(\mathbf{k}\cdot\mathbf{x})\mathbf{b}_1| \le K_1 \quad \text{and} \quad |\varphi_2(\mathbf{k}\cdot\mathbf{x})\mathbf{b}_2| \le K_2.$$

Letting $K = \max\{K_1, K_2\}$, we see that

$$|x_1(s)| \le e^{(\lambda_-)s}\left(|x_1(0)| + K_1\int_0^s |e^{-(\lambda_-)r}|\,dr + K_2\int_0^s |e^{-(\lambda_-)r}|\,dr\right)$$
$$\le e^{(\lambda_-)s}\left(|x_1(0)| + 2K\int_0^s |e^{-(\lambda_-)r}|\,dr\right)$$
$$= e^{(\lambda_-)s}\left(|x_1(0)| + \frac{2K}{\lambda_-}\right) - \frac{2K}{\lambda_-},$$

from which it follows that

$$\lim_{s\to\infty} \le -\frac{2K}{\lambda_-}.$$

It follows that the solutions of the FS have their ω-limit sets on the vertical strip S bounded by straight lines $x_1 = \pm\frac{2K}{\lambda_-}$, which turns out to be positively invariant under the flow.

Thus, we have that $\Omega(0) = \mathbb{R}^2$. by the same argument as in the last part of the proof of Proposition 12 in [3]. □

Proposition 10. *If* $d > 0$ *and* $t = 0$, *then* $\Omega(0) = \mathbb{R}^2$.

Proof. It follows by the similar argument as in the proof of Proposition 13 in [3]. □

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