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PARTIAL DIFFERENTIAL EQUATIONS AND SCALAR CURVATURE ON SEMIRIEMANNIAN MANIFOLDS (II)

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ABSTRACT. In this paper, when N is a compact Riemannian manifold, we discuss the method of using warped products to construct timelike or null future complete Lorentzian metrics on $M = [a, \infty) \times_f N$ with specific scalar curvatures.

1. Introduction

In this paper, we study the existence and nonexistence of timelike or null future complete Lorentzian warped product metric with prescribed scalar curvature functions on some Lorentzian warped product manifolds.

By the results of Kazdan and Warner [KW1, KW2, KW3], if N is a compact Riemannian n-manifold without boundary, $n \geq 3$, then N belongs to one of the following three catagories:

(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.

(B) A smooth function on N is the scalar curvature of some Rimannian metric on N if and only if the function is either identically zero or strictly negative somewhere.

(C) Any smooth function on N is the scalar curvature of some Rimannian metric on N.

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However, Kazdan and Warner [KW1, KW2, KW3] also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

Leung [L1, L2] considered the scalar curvature of some Riemannian warped product manifold and its conformal deformation of warped product metric. And also in [EJK], authors considered the existence of a nonconstant warping function on a Lorentzian warped product manifold such that the resulting warped product metric produces the constant scalar curvature when the fiber manifold has the constant scalar curvature.

Ironically, even though there exists some obstruction of positive or zero scalar curvature on a Riemannian manifold, results of [EJK], say, Theorem 3.1, Theorem 3.5 and Theorem 3.7 of [EJK] show that there exists no obstruction of positive scalar curvature on a Lorentzian warped product manifold, but there may exist some obstruction of negative or zero scalar curvature.

In this paper, when N is a compact Riemannian manifold, we discuss the method of using warped products to construct Loren tzian metrics on $M = [a, \infty) \times_f N$ with specific scalar curvatures, where a is a positive constant. In [JKLS], it is shown that if the fiber manifold N belongs to class (A) or (B), then M admits a Lorentzian metric with negative scalar curvature approaching zero near the end outside a compact set. From now on, we prove that if the fiber manifold N belongs to class (C), then M admits a timelike or null future complete Lorentzian warped metric with positive scalar curvature outside a compact set.

2. Main results

Let (N, g) be a Riemannian manifold of dimension n and let $f : [a, \infty) \to R^+$ be a smooth function, where a is a positive number. The Lorentzian warped product of N and $[a, \infty)$ with warping function f is defined to be the product manifold $([a, \infty) \times_f N, g')$ with

$$g' = -dt^2 + f^2(t)g \tag{2.1}$$

Let $R(g)$ be the scalar curvature of (N, g). Then the scalar curvature $R(t, x)$ of g' is given by the equation

$$R(t, x) = \frac{1}{f^2(t)}\{R(g)(x) + 2nf(t)f''(t) + n(n-1)|f'(t)|^2\} \tag{2.2}$$

for $t \in [a, \infty)$ and $x \in N$ (for details, cf. [DD] or [EJK]). If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (2.2) can be changed into

$$\frac{4n}{n+1}u''(t) - R(t,x)u(t) + R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{2.3}$$

In this paper, we assume that the fiber manifold N is nonempty, connected and a compact Riemannian n-manifold without boundary. Then, by Theorem 3.1, Theorem 3.5 and Theorem 3.7 in [EJK], we have the following proposition.

Proposition 2.1. *If the scalar curvature of the fiber manifold N is arbitrary constant, then there exists a nonconstant warping function $f(t)$ on $[a, \infty)$ such that the resulting Lorentzian warped product metric on $[a, \infty) \times_f N$ produces positive constant scalar curvature.*

However, the results of [EJK] show that there may exist some obstruction about the Lorentzian warped product metric with negative or zero scalar curvature even when the fiber manifold has constant scalar curvature.

Remark 2.2. By Remark 2.58 in [BE] and Corollary 5.6 in [P], if (a, b) is a finite interval and $n = 3$, then all nonspacelike geodesics are incomplete. But on $(-\infty, +\infty)$ there exists a warping function so that all non-spacelike geodesics are complete. For Theorem 5.5 in [P] implies that all timelike geodesics are future (resp. past) complete on $(-\infty, +\infty) \times_{v(t)} N$ if and only if $\int_{t_0}^{+\infty} \left(\frac{v}{1+v}\right)^{\frac{1}{2}} dt = +\infty$ (resp. $\int_{-\infty}^{t_0} \left(\frac{v}{1+v}\right)^{\frac{1}{2}} dt = +\infty$) and Remark 2.58 in [BE] implies that all null geodesics are future (resp. past) complete if and only if $\int_{t_0}^{+\infty} v^{\frac{1}{2}} dt = +\infty$ (resp. $\int_{-\infty}^{t_0} v^{\frac{1}{2}} dt = +\infty$) (cf. Theorem 4.1 and Remark 4.2 in [BEP]).

We assume that the fiber manifold N of $M = [a, \infty) \times_f N$ belongs to class (C), where a is a positive number. In this case, N admits a Riemannian metric of positive scalar curvature. If we let $u(t) = t^\alpha$, where $\alpha \in (0, 1)$ is a constant, then we have

$$R(t,x) > -\frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{t^2} \geq -\frac{4n}{n+1}\frac{1}{4}\frac{1}{t^2}, \quad t > a.$$

By the similar proof like as Theorem 2.4 in [JKLS], we have the following:

Theorem 2.3. *If $R(g)$ is positive, then there is no positive solution to equation (2.3) with*

$$R(t) \leq -\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \textit{for} \quad t \geq t_0,$$

where $c > 1$ and $t_0 > a$ are constants.

If N belongs to (C), then any smooth function on N is the scalar curvature of some Riemannian metric. So we can take a Riemannian metric g_1 on N with scalar curvature $R(g_1) = \frac{4n}{n+1}k^2$, where k is a positive constant. Then equation (2.3) becomes

$$\frac{4n}{n+1}u''(t) + \frac{4n}{n+1}k^2 u(t)^{1-\frac{4}{n+1}} - R(t,x)u(t) = 0. \tag{2.4}$$

If $R(t,x)$ is the function of only t-variable, then we have the following theorem.

Theorem 2.4. *Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a positive function such that*

$$b \geq R(t) \geq \frac{4n}{n+1}\frac{C}{t^\alpha} \quad \textit{for} \quad t \geq t_0,$$

where $t_0 > a$, $\alpha < 2$, C and b are positive constants. Then equation (2.4) has a positive solution on $[a,\infty)$ and the resulting Lorentzian warped product metric is a future geodesically complete metric of positive scalar curvature outside a compact set.

Proof. We let $u_+ = t^m$ and $u_- = c_0$ as a small positive constant, where m are positive numbers. If we take m large enough so that $m\frac{4}{n+1} > 2$, then we have, $t \geq t_0$ for some large t_0,

$$\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} - R(t)u_+(t) \\
&\leq \frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} - \frac{4n}{n+1}\frac{C}{t^\alpha}u_+(t) \\
&= \frac{4n}{n+1}t^m\left[\frac{m(m-1)}{t^2} + \frac{k^2}{t^{m\frac{4}{n+1}}} - \frac{C}{t^\alpha}\right] \\
&\leq 0,
\end{aligned}$$

which is possible for large fixed m since $\alpha < 2$. And since the exponent $1 - \frac{4}{n+1}$ is less than 1 and $R(t)$ is a bounded function,

$$\frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}k^2 u_-(t)^{1-\frac{4}{n+1}} - R(t)u_-(t) \geq 0.$$

Since $t > t_0 > a > 0$, we can take the lower solution $u_-(t) = c_0$ so that $0 < u_-(t) < u_+(t)$. So by the upper and lower solution method, we obtain a positive solution $u(t) = f(t)^{\frac{n+1}{2}}$ such that $0 < u_-(t) \le u(t) \le u_+(t)$. Hence

$$\int_{t_0}^{+\infty} \left(\frac{f(t)}{1+f(t)}\right)^{\frac{1}{2}} dt = \int_{t_0}^{+\infty} \left(\frac{u(t)^{\frac{2}{n+1}}}{1+u(t)^{\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt \ge \int_{t_0}^{+\infty} \left(\frac{c_0^{\frac{2}{n+1}}}{1+c_0^{\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt = +\infty$$

and

$$\int_{t_0}^{+\infty} f(t)^{\frac{1}{2}} dt = \int_{t_0}^{+\infty} u(t)^{\frac{1}{n+1}} dt \ge \int_{t_0}^{+\infty} c_0^{\frac{1}{n+1}} dt = +\infty,$$

which, by Remark 2.2, implies that the resulting warped product metric is a future geodesically complete one. □

Theorem 2.5. *Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a positive function such that*

$$b \ge R(t) \ge \frac{C}{t^2} \quad \text{for} \quad t \ge t_0,$$

where $t_0 > a$, b and C are positive constants. If $C > n(n-1)$, then equation (2.4) has a positive solution on $[a,\infty)$ and the resulting Lorentzian warped metric is a future nonspacelike geodesically complete metric of positive scalar curvature outside a compact set.

Proof. In case $C > n(n-1)$, we may take $u_+ = C_+ t^{\frac{n+1}{2}}$, where C_+ is a positive constant. Then

$$\begin{aligned}&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} - R(t)u_+(t)\\ &\le C_+\frac{4n}{n+1}t^{\frac{n-3}{2}}\left[\frac{n^2-1}{4} + k^2 C_+^{-\frac{4}{n+1}} - \frac{n+1}{4n}C\right] \le 0,\end{aligned}$$

which is possible if we take C_+ to be large enough since $\frac{(n+1)(n-1)}{4} - \frac{n+1}{4n}C < 0$. And since the exponent $1 - \frac{4}{n+1}$ is less than 1 and $R(t)$ is a bounded function, we can take $u_-(t) = c_0$ to be a small positive constant on $[a,\infty)$. In this case, we also obtain a positive solution as in Theorem 2.4. Hence

$$\int_{t_0}^{+\infty} \left(\frac{f(t)}{1+f(t)}\right)^{\frac{1}{2}} dt \int_{t_0}^{+\infty} \left(\frac{u(t)^{\frac{2}{n+1}}}{1+u(t)^{\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt \ge \int_{t_0}^{+\infty} \left(\frac{c_0^{\frac{2}{n+1}}}{1+c_0^{\frac{2}{n+1}}}\right)^{\frac{1}{2}} dt = +\infty$$

and

$$\int_{t_0}^{+\infty} f(t)^{\frac{1}{2}} dt = \int_{t_0}^{+\infty} u(t)^{\frac{1}{n+1}} dt \geq \int_{t_0}^{+\infty} c_0^{\frac{1}{n+1}} dt = +\infty,$$

which, by Remark 2.2, implies that the resulting warped product metric is a future nonspacelike geodesically complete one. □

Remark 2.6. By Theorem 3.4 and Corollary 3.5 in [J], the result in Theorem 2.5 is almost sharp as we can get as close to $\frac{n(n-1)}{t^2}$ as possible. For example, let $R(g) = \frac{4n}{n+1}k^2$ and $f(t) = t \ln t$ for $t > a$. Then we have

$$R = \frac{1}{t^2}\left[\frac{4n}{n+1}\frac{k^2}{(\ln t)^2} + \frac{2n}{\ln t} + n(n-1)(1+\frac{1}{\ln t})^2\right]$$

which converges to $\frac{n(n-1)}{t^2}$ as t goes to ∞.

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