J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 8, Number 2 (Novemer 2001), Pages 137–143

EXISTENCE, UNIQUENESS AND NORM ESTIMATE OF SOLUTIONS FOR THE NONLINEAR DELAY INTEGRO-DIFFERENTIAL SYSTEM

JONG SEO PARK AND YOUNG CHEL KWUN

ABSTRACT. In this paper, we study the existence, uniqueness and norm estimate of solutions for the nonlinear delay integro-differential system.

1. INTRODUCTION

The existence for solutions of evolution equation with the nonlocal conditions in Banach space has been studied first by Byszewski [1].

In this paper, we study the existence, uniqueness and norm estimate of solutions for the following nonlinear delay integro-differential system with nonlocal initial condition :

$$(1.1)\qquad \begin{cases} \dfrac{dx(t)}{dt} = Ax(t) + f\big(t, x_t, \displaystyle\int_0^t k(t,s,x_s)ds\big), & t \in (0,T]; \\ x(t) + g(x_{t_1}, \cdots, x_{t_p})(t) = \phi(t), & t \in [-h,0], \end{cases}$$

where $0 < t_1 < \cdots < t_p \le T$ ($p \in \mathbb{N}$), bounded linear operator A is the infinitesimal generator of C_0 semigroup on a Banach space.

$C([-h,0] : X)$ is a Banach space of all continuous functions from an interval $[-h,0]$ to X with the norm defined by supremum,

$$f : [0,T] \times C([-h,0] : X) \times X \to X,$$
$$g : \big[C([-h,0] : X)\big]^p \to C([-h,0] : X),$$
$$k : [0,T] \times [0,T] \times C([-h,0] : X) \to X$$

are given nonlinear functions and ϕ is a initial function. If a function x is continuous from $[-h,0] \cup [0,T]$ to X, then x_t is an element in $C([-h,0] : X)$ which has pointwise

Received by the editors May 2, 2001, and in revised form June 13, 2001.
2000 *Mathematics Subject Classification.* 93C10, 93C20.
Key words and phrases. existence, uniqueness, norm estimate, nonlocal initial condition.

©2001 Korea Soc. Math. Educ.

definition:

$$x_t(\theta) = x(t+\theta) \text{ for } \theta \in [-h, 0],\ t \in [0, T].$$

2. Existence, Uniqueness and Norm Estimate Results

We consider the following integral equation

$$(2.1)\quad \begin{cases} x(t) = S(t)\{\phi(0) - g(x_{t_1}, \cdots, x_{t_p})(0)\} \\ \qquad\qquad + \int_0^t S(t-s) f(s, x_s, \int_0^s k(s, \tau, x_\tau) d\tau) ds, & t \in [0, T], \\ x(t) + g(x_{t_1}, \cdots, x_{t_p})(t) = \phi(t), & t \in [-h, 0]. \end{cases}$$

Then the continuous solution $x(t) \in C([-h, T] : X)$ of (2.1) is called the *mild solution* of (1.1).

Let X be a Banach space with the norm $\|\cdot\|$. We shall make the following assumptions of f, k, g and a C_0 semigroup $S(t)$.

(H1) There exists a constant L such that

$$\|f(t, \varphi_1, \psi_1) - f(t, \varphi_2, \psi_2)\| \le L(\|\varphi_1 - \varphi_2\|_{C([-h,T]:X)} + \|\psi_1 - \psi_2\|_X),$$

for $\varphi_1, \varphi_2 \in C([-h, 0] : X)$, $\psi_1, \psi_2 \in X$, $t \in [0, T]$ and $f(t, 0, 0) \equiv 0$.

(H2) The nonlinear function

$$k : [0, T] \times [0, T] \times C([-h, 0] : X) \to X$$

satisfies a Lipschitz condition

$$\|k(t, s, \varphi_1) - k(t, s, \varphi_2)\| \le L_1 \|\varphi_1 - \varphi_2\|_{C([-h,T]:X)},$$

where $\varphi_1, \varphi_2 \in C([-h, 0] : X)$, $t, s \in [0, T]$, L_1 is constant and $k(t, s, 0) \equiv 0$.

(H3) The nonlinear function

$$g : [C([-h, 0] : X)]^p \to C([-h, 0] : X)$$

satisfies a Lipschitz condition

$$\|g(x_{t_1}, \cdots, x_{t_p})(s) - g(\tilde{x}_{t_1}, \cdots, \tilde{x}_{t_p})(s) \| \le K \|x_t - \tilde{x}_t\|_{C([-h,0]:X)},$$

where x_t, $\tilde{x}_t \in C([-h, 0] : X)$, $s \in [-h, 0]$ and $K > 0$ is constant.

(H4) $\phi \in C([-h, 0] : X)$.

(H5) $\|S(t)\| \le M$.

Now, we will prove the following theorem.

Theorm 2.1. *Assume that the hypotheses* (H1)–(H5) are satisfied and

$$M\{K + LT(1 + L_1T)\} < 1,$$

then nonlocal Cauchy problem (1.1) has unique mild solution.

Proof. Define the operator $\mathcal{F} : C([-h,T] : X) \to C([-h,T] : X)$ by

$$(\mathcal{F}x_t)(0) = \begin{cases} \phi(t) - g(x_{t_1}, \cdots, x_{t_p})(t), & t \in [-h, 0], \\ S(t)\{\phi(0) - g(x_{t_1}, \cdots, x_{t_p})(0)\} \\ \qquad + \int_0^t S(t-s) f(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau)ds, & t \in [0,T], \end{cases}$$

for $x_t \in C([-h,T] : X)$.

We will prove that $\mathcal{F}$ is contractive mapping defined by $(\mathcal{F}x_t)(\theta) = (\mathcal{F}x_{t+\theta})(0)$. If $x_t,\ y_t \in C([-h,T] : X)$ and $t + \theta \in [-h, 0]$,

$$\begin{aligned} (\mathcal{F}x_t)(\theta) - (\mathcal{F}y_t)(\theta) &= (\mathcal{F}x_{t+\theta})(0) - (\mathcal{F}y_{t+\theta})(0) \\ &= g(x_{t_1}, \cdots, x_{t_p})(t+\theta) - g(y_{t_1}, \cdots, y_{t_p})(t+\theta). \end{aligned} \tag{2.2}$$

And if $x_t,\ y_t \in C([-h,T] : X)$ and $t + \theta \in [0, T]$, then

$$\begin{aligned} &(\mathcal{F}x_t)(\theta) - (\mathcal{F}y_t)(\theta) \\ &= (\mathcal{F}x_{t+\theta})(0) - (\mathcal{F}y_{t+\theta})(0) \\ &= S(t+\theta)\{g(x_{t_1}, \cdots, x_{t_p})(0) - g(y_{t_1}, \cdots, y_{t_p})(0)\} \\ &\quad + \int_0^{t+\theta} S(t+\theta-s) \left\{ f(s, x_s, \int_0^s k(s,\tau,x_\tau)d\tau) - f(s, y_s, \int_0^s k(s,\tau,y_\tau)d\tau) \right\} ds. \end{aligned} \tag{2.3}$$

From (2.2) and (H3),

$$\|(\mathcal{F}x_t)(\theta) - (\mathcal{F}y_t)(\theta)\| \le K\|x_t - y_t\|_{C([-h,T]:X)}.$$

Hence

$$\begin{aligned} \|(\mathcal{F}x_t) - (\mathcal{F}y_t)\|_{C([-h,T]:X)} &= \sup_{-h \le \theta \le 0} \|(\mathcal{F}x_t)(\theta) - (\mathcal{F}y_t)(\theta)\| \\ &\le K\|x_t - y_t\|_{C([-h,T]:X)}. \end{aligned}$$

From 2.3 and (H1)–(H5),

$$\begin{aligned}
&\|(\mathcal{F}x_t)(\theta) - (\mathcal{F}y_t)(\theta)\| \\
&\le \|S(t+\theta)\|\|g(x_{t_1},\cdots,x_{t_p})(0) - g(y_{t_1},\cdots,y_{t_p})(0)\| \\
&\quad + \int_0^{t+\theta} \|S(t+\theta-s)\| \left\| f(s,x_s,\int_0^s k(s,\tau,x_\tau)d\tau) - f(s,y_s,\int_0^s k(s,\tau,y_\tau)d\tau)\right\| ds \\
&\le MK\|x_t - y_t\|_{C([-h,T]:X)} \\
&\qquad + M\int_0^{t+\theta} L\left\{\|x_s - y_s\|_{C([-h,T]:X)} + \int_0^s L_1\,|x_\tau - y_\tau\|_{C([-h,T]:X)}d\tau\right\} ds \\
&\le MK\|x_t - y_t\|_{C([-h,T]:X)} + ML\int_0^{t+\theta}\{1 + L_1(t+\theta)\}\,\|x_s - y_s\|_{C([-h,T]:X)}\,ds \\
&\le MK\|x_t - y_t\|_{C([-h,T]:X)} + ML\{1 + L_1(t+\theta)\}(t+\theta)\,\|x_t - y_t\|_{C([-h,T]:X)}\int_0^{t+\theta} ds \\
&\le M\{K + LT(1 + L_1T)\}\|x_t - y_t\|_{C([-h,T]:X)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\mathcal{F}x_t - \mathcal{F}y_t\|_{C([-h,T]:X)} &= \sup_{-h\le\theta\le 0}\|\mathcal{F}x_t(\theta) - \mathcal{F}y_t(\theta)\| \\
&\le M\{K + LT(1 + L_1T)\}\|x_t - y_t\|_{C([-h,T]:X)}.
\end{aligned}$$

Since $M\{K + LT(1 + L_1T)\} < 1$, $\mathcal{F}$ is a contractive mapping on $C([-h,T]:X)$. Consequently, an unique fixed point of $\mathcal{F}$ on $C([-h,T]:X)$ is a unique mild solution of (1.1). □

Next theorem is characteristic for the continuous dependence of the nonlinear functional integro-differential system (1.1) with the classical initial condition.

Theorm 2.2. *Suppose that the hypotheses* (H1)–(H5) are holds and

$$M\{K + LT(1 + L_1T)\} < 1.$$

Then for each $\phi_1,\ \phi_2 \in C([-h,0]:X)$ and mild solution $x_t^1,\ x_t^2$ of the equations

$$\text{(2.4)}\qquad \begin{cases} \dfrac{dx^i(t)}{dt} = Ax^i(t) + f(t,x_t^i,\displaystyle\int_0^t k(t,s,x_s^i)ds), & t\in[0,T] \\ x^i(t) + g(x_{t_1}^i,\cdots,x_{t_p}^i)(t) = \phi_i(t), & t\in[-h,0],\ i = 1,\ 2, \end{cases}$$

the following inequality is established

$$\text{(2.5)}\qquad \|x_t^1 - x_t^2\|_{C([-h,T]:X)} \le \frac{M}{1-MK}\exp\frac{ML(1+L_1T)T}{1-MK}\,\|\phi_1 - \phi_2\|_{C([-h,0]:X)}.$$

Proof. Let $\phi_i \in C([-h,0]:X)$ $(i=1,\ 2)$ and x_t^i $(i=1,\ 2)$ be the mild solution of (2.4). For $t+\theta\in[0,T]$,

$$\begin{aligned}
&x_t^1(\theta)-x_t^2(\theta)\\
&= S(t+\theta)\{\phi_1(0)-\phi_2(0)\} - S(t+\theta)\{g(x_{t_1}^1,\cdots,x_{t_p}^1)(0)-g(x_{t_1}^2,\cdots,x_{t_p}^2)(0)\}\\
&\quad+\int_0^t S(t+\theta-s)\Big\{f(s,x_s^1,\int_0^s k(s,\tau,x_\tau^1)d\tau)-f(s,x_s^2,\int_0^s k(s,\tau,x_\tau^2)d\tau)\Big\}ds
\end{aligned}\tag{2.6}$$

and for $t+\theta\in[-h,0]$,

$$\begin{aligned}
&x_t^1(\theta)-x_t^2(\theta)\\
&=\phi_1(t+\theta)-\phi_2(t+\theta)+g(x_{t_1}^2,\cdots,x_{t_p}^2)(t+\theta)-g(x_{t_1}^1,\cdots,x_{t_p}^1)(t+\theta).
\end{aligned}\tag{2.7}$$

From 2.6 and (H1)–(H5),

$$\begin{aligned}
&\|x_t^1(\theta)-x_t^2(\theta)\|\\
&\le M\|\phi_1-\phi_2\|_{C([-h,0]:X)}+MK\|x_t^1-x_t^2\|_{C([-h,T]:X)}\\
&\qquad+M\int_0^{t+\theta}L\left(\|x_s^1-x_s^2\|_{C([-h,T]:X)}\int_0^s L_1\|x_\tau^1-x_\tau^2\|_{C([-h,T]:X)}d\tau\right)ds\\
&\le M\|\phi_1-\phi_2\|_{C([-h,0]:X)}+MK\|x_t^1-x_t^2\|_{C([-h,T]:X)}\\
&\qquad+ML\big(1+L_1(t+\theta)\big)\int_0^{t+\theta}\|x_s^1-x_s^2\|_{C([-h,T]:X)}ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|x_t^1-x_t^2\|_{C([-h,T]:X)}\\
&=\sup_{-h\le\theta\le 0}\|x_t^1(\theta)-x_t^2(\theta)\|\\
&\le M\|\phi_1-\phi_2\|_{C([-h,0]:X)}+MK\|x_t^1-x_t^2\|_{C([-h,T]:X)}\\
&\qquad+ML(1+L_1T)\int_0^T\|x_s^1-x_s^2\|_{C([-h,T]:X)}ds.
\end{aligned}\tag{2.8}$$

From 2.7 and (H3)–(H4),

$$\|x_t^1(\theta)-x_t^2(\theta)\|\le\|\phi_1-\phi_2\|_{C([-h,0]:X)}+K\|x_t^1-x_t^2\|_{C([-h,T]:X)}.$$

Thus

$$\begin{aligned}
\|x_t^1-x_t^2\|_{C([-h,T]:X)}&=\sup_{-h\le\theta\le 0}\|x_t^1(\theta)-x_t^2(\theta)\|\\
&\le\|\phi_1-\phi_2\|_{C([-h,0]:X)}+K\|x_t^1-x_t^2\|_{C([-h,T]:X)}.
\end{aligned}\tag{2.9}$$

Since $M \geq 1$ and $MK < 1$, then 2.8 and 2.9 imply that

$$\|x_t^1 - x_t^2\|_{C([-h,T]:X)} \leq \frac{M}{1-MK}\|\phi_1 - \phi_2\|_{C([-h,0]:X)} + \frac{ML(1+L_1T)}{1-MK}\int_0^T \|x_s^1 - x_s^2\|_{C([-h,T]:X)}ds.$$

By Gronwall's inequality,

$$\|x_t^1 - x_t^2\|_{C([-h,T]:X)} \leq \frac{M}{1-MK}\|\phi_1 - \phi_2\|_{C([-h,0]:X)} \exp\left(\frac{ML(1+L_1T)T}{1-MK}\right). \quad \square$$

Remark 2.1. Let $0 < t_1 < \cdots < t_p \leq T$ $(p \in N)$, Theorems 2.1 and 2.2 can be employed the following g defined by

$$g(x_{t_1}, \cdots, x_{t_p})(s) = \sum_{k=1}^{p} c_k x(t_k + s),$$

where $x \in C([-h,T]:X)$, $s \in [-h,0]$ and c_k $(k = 1, \cdots, p)$ is constant satisfying

$$M\left\{\sum_{k=1}^{p} |c_k| + LT(1 + L_1T)\right\} < 1. \tag{2.10}$$

Remark 2.2. Let $0 < t_1 < \cdots < t_p$ and ϵ_k $(k = 1, \cdots, p)$ is constant such that $0 < t_1 - \epsilon_1$ and $t_{k-1} < t_k - \epsilon_k$ $(k = 2, \cdots, p)$. If the nonlinear function g is defined by

$$g(x_{t_1}, \cdots, x_{t_p})(s) = \sum_{k=1}^{p} \frac{c_k}{\epsilon_k}\int_{t_k-\epsilon_k} x(\tau + s)d\tau,$$

where $x \in C([-h,T]:X)$, $s \in [-h,0]$ and c_k $(k = 1, \cdots, p)$ is constant satisfying 2.10. For $s \in [-h,0]$, since

$$\begin{aligned}\|g(x_{t_1}, \cdots, x_{t_p})(s) - g(y_{t_1}, \cdots, y_{t_p})(s)\| &= \left\|\sum_{k=1}^{p} \frac{c_k}{\epsilon_k}\int_{t_k-\epsilon_k} \big(x(\tau+s) - y(\tau+s)\big)d\tau\right\| \\ &\leq \left(\sum_{k=1}^{p} |c_k|\right)\|x_t - y_t\|_{C([-h,T]:X)},\end{aligned}$$

Theorems 2.1 and 2.2 can be employed the function g.

ACKNOWLEDGMENTS

The authors are grateful to the referee for his careful reading and noting errors in the original manuscript.

REFERENCES

1. L. Byszewski: Theorems about the existence and uniqueness of solutions a semilinear evolution nonlocal Cauchy problem. *J. Math. Anal. Appl.* **162** (1991), no. 2, 494–505. MR **92m**:35005
2. L. Byszewski and V. Lakshmikantham: Theorem about the existence and uniqueness of a solution of a nonlocal abstrct Cauchy problem in a Banach space. *Appl. Anal.* **40** (1990), 11–19. MR **92h**:34121
3. D. Fujiwara: Concrete characterizations of the domains of functional powers of source elliptic differential operators of the second order. *Proc. Japan Acad.* **43** (1967), 82–86. MR **35**#7170
4. D. Henry: *Geometric theory of semilinear parabolic equations.* Springer Lecture Notes in Math., 840, Springer-Verlag, Berlin-New York, 1981. MR **83j**:35084
5. S. Ilamaran and K. Balachandran: Existence and uniqueness of mild and strong solutions of a integrodifferential equation with nonlocal condition. *Libertas Math.* **15** (1995), 119–127. MR **96f**:34117
6. B. G. Pachpatte: A note on Gronwall-Bellman inequality. *J. Math. Anal. Appl.* **44** (1973), 758–762. MR **49** #501
7. J. S. Park and Y. C. Kwun: Controllability of nonlinear delay parabolic equations with nonlocal initial condition under boundary input. *J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math.* **7** (2000), no. 2, 101–113. CMP (2001:09)1 816 556
8. H. K. Pathak: Application of fixed point theorems to abstract volterra integrodifferential equations. *Riv. Mat. Univ. Parma (5)* **3** (1994), 193–202. MR **96h**:45003
9. A. Pazy: *Semigroups of linear operators and applications to partial differential equations.* Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983. MR **85g**:47061

(J. S. PARK) DEPARTMENT OF MATHEMATICS EDUCATION, CHINJU NATIONAL UNIVERSITY OF EDUCATION, 380 SINAN-DONG, JINJU, GYEONGNAM 540-742, KOREA

E-mail address: parkjs@cue.ac.kr

(Y. C. KWUN) DEPARTMENT OF MATHEMATICS, DONG-A UNIVERSITY, 840 HADAN-DONG, SAHA-GU, BUSAN 604-714, KOREA

E-mail address: yckwun@daunet.donga.ac.kr