J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
7(2000), no. 1, 27–32

CERTAIN REAL QUADRATIC FIELDS WITH CLASS NUMBERS 1, 3 AND 5

JOONG SOO PARK

ABSTRACT. The quadratic fields generated by $x^2 = ax + 1$ $(a \geq 1)$ are studied. The regulators are relatively small and are known at once. The class numbers are relatively large and easy to compute. We shall find all the values of p, where $p = a^2 + 4$ is a prime in $\mathbb{Z}$, such that $\mathbb{Q}(\sqrt{p})$ has class numbers 1, 3 and 5.

1. Introduction

Siegel [10] proved that

$$\frac{\log h_K}{\log |\mathrm{disc}(K)|^{1/2}} \to 1$$

as $|\mathrm{disc}(K)| \to \infty$ for imamginary quadratic fields. It follows that there are at most finitely many d for which $\mathbb{Q}(\sqrt{-d})$ has class number below a fixed bound (cf. [1], [2], [7], [11], [12], [14]). For real quadratic fields, Gauss conjectured that there are infinitely many fields $\mathbb{Q}(\sqrt{d})$ has class number 1. This is still an open problem (cf. [3], [4], [6], [15]).

Shanks [9] introduced the simplest cubic fields generated by $x^3 = ax^2 + (a+3)x + 1$ where $a^2 + 3a + 9$ is a prime p in $\mathbb{Z}$. Particularly, in §3, he mentioned the quadratic analogue of the simplest cubic fields. He obtained the class numbers of certain quadratic fields with small discriminants.

In this paper, we let K be the quadratic fields generated by $x^2 = ax + 1$. Let $a \geq 1$ be an integer such that $a^2 + 4$ is a prime p in $\mathbb{Z}$. Then $K = \mathbb{Q}(\sqrt{p})$. Let h be the class number of K. In view of Gauss conjecture, it is interesting to find all values of p, $p = a^2 + 4$, such that the quadratic field $\mathbb{Q}(\sqrt{p})$ has class number 1. We

Received by the editors December 8, 1999, and in revised form March 18, 2000.
2000 *Mathematics Subject Classification.* 11R16, 11R29.
Key words and phrases. quadratic fields, class number
This paper was supported by Woosuk University Research Fund in 2000.

shall go further and find all the values of p, where $p = a^2 + 4$ is a prime, such that $\mathbb{Q}(\sqrt{p})$ has class numbers 1, 3 and 5.

2. Quadratic fields

Let K be a quadratic field generated by $x^2 = ax + 1$ so that $K = \mathbb{Q}(\sqrt{a^2+4})$. Assume that $a \geq 1$ and that a^2+4 is a prime p in $\mathbb{Z}$. Since $p = a^2+4 \equiv 1 \pmod 4$, the ring O_K of algebraic integers in K is $\mathbb{Z} \oplus \mathbb{Z}(\frac{1+\sqrt{p}}{2})$ and the discriminant disc(K) of K is p. Let $\theta = \frac{1}{2}(\sqrt{p}+a)$. Then θ is a root of $x^2 - ax - 1 = 0$. Then $\theta(\theta - a) = 1$, thus θ is a unit. Also, it is a (the) fundamental unit. Note that the norm of θ is -1 and the regulator R_K of K is $\log\theta$.

In the Dedekind class number formula $h = \frac{\rho}{k}$, we have $k = \frac{2\log\theta}{\sqrt{p}}$ and $\rho = \lim_{s\to 1} \frac{\zeta_K(s)}{\zeta(s)}$. The Dedekind zeta function $\zeta_K(s)$ on K is expressed by

$$\zeta_K(s) = \prod_q (1 - \frac{1}{p^{f_q s}})^{-r_q}$$

where q runs through all primes in $\mathbb{Z}$ and f_q is the degree of inertia of q and r_q is the number of primes lying over q into O_K. As we know that the Riemann zeta function $\zeta(s)$ has the expression that

$$\zeta(s) = \prod_q (1 - \frac{1}{q^s})^{-1}$$

where q runs through all primes in $\mathbb{Z}$.

Since $\lim_{s\to 1} \frac{\zeta_K(s)}{\zeta(s)}$ converges,

$$\lim_{s\to 1} \frac{\zeta_K(s)}{\zeta(s)} = \frac{\prod_q (1 - \frac{1}{q^{f_q s}})^{-r_q}}{\prod_q (1 - \frac{1}{q})^{-1}} = \prod_q f(q) \qquad (*)$$

where q runs through all primes in $\mathbb{Z}$ and

$$f(q) = \begin{cases} 1 & \text{if } q \text{ ramifies in } K, \\ \frac{q}{q-1} & \text{if } q \text{ splits in } K, \\ \frac{q}{q+1} & \text{if } q \text{ inerts in } K. \end{cases}$$

Let χ be a Dirichlet character modulo m. The Dirichlet L-function $L(s,\chi)$ associated to χ is the function given by

$$L(s,\chi) = \prod_q (1 - \frac{\chi(q)}{q^s})^{-1}$$

where q runs through all primes in $\mathbb{Z}$ which does not divide m. Rewriting the Dedekind zeta function, we have

$$\begin{aligned}\zeta_K(s) &= \prod_q (1-\frac{1}{q^{f_q s}})^{-r_q} \\ &= \prod_{q_i}(1-\frac{1}{q_i{}^s})^{-2}\prod_{q_j}(1-\frac{1}{q_j{}^{2s}})^{-1}\prod_{q_k}(1-\frac{1}{q_k{}^s})^{-1} \\ &= \prod_q(1-\frac{1}{q^s})^{-1}\{\prod_{q_i}(1-\frac{1}{q_i{}^s})^{-1}\prod_{q_j}(1-\frac{(-1)}{q_j{}^s})^{-1}\} \\ &= \zeta(s)\prod_q(1-\frac{\chi_p(q)}{q^s})^{-1}\end{aligned}$$

where q_i, q_j, q_k run through all splitting, inerting, ramifying primes in $\mathbb{Z}$ and q runs through all primes in $\mathbb{Z}$ respectively, and $\chi_p(q)$ is a Dirichlet character modulo p such that

$$\chi_p(q) = \begin{cases} 1 & \text{if } q \text{ splits in } K, \text{ i.e., } (\frac{q}{p}) = 1, \\ -1 & \text{if } q \text{ inerts in } K, \text{ i.e., } (\frac{q}{p}) = -1, \\ 0 & \text{if } q = p. \end{cases}$$

Note that χ_p is the Legendre symbol modulo p.

Theorem 1 (Tatuzawa [5, 13, 15]). *For any positive c satisfing* $\frac{1}{2} > c > 0$, *let d be any positive integer such that* $d \geq \max\{e^{\frac{1}{c}}, e^{11.2}\}$, *and let* χ_d *be any non-principal primitive character modulo d. Then, it holds with one possible exception of d*

$$L(1,\chi_d) > (0.655)\frac{c}{d^c}$$

where $L(1,\chi_d)$ *is the value of the Dirichlet L-function* $L(s,\chi_d)$ *at* $s = 1$.

If we put $s = \frac{1}{c}$ in Theorem 1 and take χ_p to be the Legendre symbol modulo p, then we have the following corollary.

Corollary 2. *For any* $s \geq 11.2$, *we have*

$$L(1,\chi_p) > (\frac{0.655}{s})p^{-\frac{1}{s}}$$

where χ_p *is the Legendre symbol modulo p.*

With this lower bound of the Dirichlet L-function we have the following a classification of certain real quadratic fields.

Theorem 3. *Let $K = \mathbb{Q}(\sqrt{p})$ where p is a prime in $\mathbb{Z}$ of the form a^2+4, $a \geq 1$. Then possibly with one exception we have;*

(1) *$h = 1$ only for $p = 5, 13, 29, 53, 173, 293$.*
(2) *$h = 3$ only for $p = 229, 733, 1229, 1373, 2213, 4493, 5333, 9413$.*
(5) *$h = 5$ only for $p = 1093, 3253, 7229, 10613, 13229, 18773, 27893, 37253$.*

Proof. In the Dedekind class number formula,

$$\begin{aligned} h &= \frac{\sqrt{p}}{2\log(\frac{\sqrt{p}+\sqrt{p-4}}{2})} L(1,\chi_p) \\ &> \frac{\sqrt{p}}{2\log\sqrt{p}}(\frac{0.655}{s})p^{-\frac{1}{s}} \\ &= \frac{\sqrt{p}}{\log p}(\frac{0.655}{s})p^{-\frac{1}{s}} \\ &= (\frac{0.655}{s})\frac{p^{\frac{s-2}{2s}}}{\log p}. \end{aligned}$$

In Corollary 2, if we take $p \geq e^{14}$ and $s = 14$, then we have

$$h > (\frac{0.655}{14})\frac{e^6}{14} = 1.34819\cdots > 1.$$

For the class number h greater than 4 (resp. 6), take $p \geq e^{17}$ and $s = 17$ (resp. $p \geq e^{18}$ and $s = 18$).

Table I

a	p	h	a	p	h	a	p	h
1	5	1	65	4229	7	155	24029	9
3	13	1	67	4493	3	163	26573	9
5	29	1	73	5333	3	167	27893	5
7	53	1	85	7229	5	177	31333	19
13	173	1	87	7573	9	183	33493	9
15	229	3	95	9029	7	193	37253	5
17	293	1	97	9413	3	203	41213	7
27	733	3	103	10613	5	207	42853	15
33	1093	5	115	13229	5	215	46229	13
35	1229	3	117	13693	15	217	47093	9
37	1373	3	125	15629	9	233	54293	9
45	2029	7	135	18229	19	235	55229	15
47	2213	3	137	18773	5	243	59053	25
57	3253	5	147	21613	13	245	60029	13

Table II

a	p	h	a	p	h	a	p	h
253	64013	9	477	227533	27	755	570029	33
255	65029	27	483	233293	45	763	582173	25
265	70229	19	487	237173	11	767	588293	23
267	71293	15	495	245029	63	785	616229	35
275	75629	21	503	253013	15	787	619373	25
277	76733	7	507	257053	31	795	632029	45
287	82373	13	533	284093	17	805	648029	37
293	85853	11	547	299213	15	813	660973	51
303	91813	23	573	328333	33	823	677333	21
307	94253	9	577	332933	15	827	683933	19
313	97973	13	593	351653	31	837	700573	61
317	100493	13	605	366029	31	845	714029	43
347	120413	11	607	368453	19	847	717413	29
357	127453	33	613	375773	13	853	727613	23
373	139133	15	615	378229	49	883	779693	21
375	140629	25	623	388133	23	897	804613	51
385	148229	23	663	439573	41	903	815413	91
403	162413	15	665	442229	35	905	819029	39
407	165653	13	667	444839	23	935	874229	43
423	178933	29	677	458333	21	945	893029	99
425	180629	21	685	469229	27	953	908213	27
427	182333	17	703	494213	25	963	927373	39
435	189229	43	707	499853	33	967	935093	25
447	199813	35	713	508373	17	973	946733	27
453	205213	27	717	514093	31	983	966293	33
455	207029	33	743	552053	15	993	986053	55
463	214373	17	745	555029	27	997	994013	25
475	225629	17	753	567013	29			

Table III

p	h	p	h
$10^6 \leq p < 10 \cdot 10^6$	$27 \leq h$	$20 \cdot 10^6 \leq p < 40 \cdot 10^6$	$79 \leq h$
$10 \cdot 10^6 \leq p < 20 \cdot 10^6$	$55 \leq h$	$40 \cdot 10^6 \leq p < 70 \cdot 10^6$	$97 \leq h$

where $e^{18} \approx 6.566 \cdot 10^7 < 70 \cdot 10^6$.

From Tables I, II and III, we have the desired results. $\square$

Remark.

(1) For the class number of real quadratic field $\mathbb{Q}(\sqrt{p})$, by the Gauss genus theory for quadratic forms, we have that the class number h is always odd.

(2) For the calculation of class numbers (Tables I, II and III), we used (*) of Section 2 and the fact that the class number h is odd. We programmed with Mathematica 3.0.

(3) In [8], we see that the number of primes p of the form $a^2 + 4$, $p \leq 70 \cdot 10^6$, is 744.

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Department of Mathematics, Woosuk University, Wanju, Chonbuk 565-701, Korea
E-mail address: jspark@core.woosuk.ac.kr