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A CHARACTERIZATION OF ASYMPTOTIC STABILITY IN DYNAMICAL POLYSYSTEMS

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ABSTRACT. We show that a compact subset M of X is asymptotically stable if and only if a strict Lyapunov function of M exists.

1. Introduction and Preliminaries

The subject of Lyapunov functions constitutes a control theme in the theory of differential equations or more generally dynamical systems. The basic feature of the stability theory on Lyapunov functions characterizes stability and asymptotic stability of a given set in terms of a nonnegative scalar function defined on a neighborhood of the given set.

The purpose of this paper is to explore that a compact set M is asymptotically stable if and only if it has a strict Lyapunov function defined on a neighborhood of M in dynamical polysystems.

Next we recall some basic concepts and notation from [7]. Throughout this paper X will be locally compact metric space unless otherwise stated and $\mathbb{R}^+$ the set of nonnegative real numbers. A dynamical system on X is a continuous map in $\pi : X \times \mathbb{R} \to X$ with the following properties:

(1) $\pi(x, 0) = x$ for all $x \in X$, and
(2) $\pi(\pi(x, s), t) = \pi(x, s + t)$ for all $x \in X$ and $s, t \in \mathbb{R}$.

Using a term introduced by Lobry [4], we shall call a family of dynamical systems $\{\pi_i | \, i \in I\}$ a *dynamical polysystem* on X, where I is an arbitrary set of indices.

Let $\{\pi_i | \, i \in I\}$ be a dynamical polysystem on X. We 2^X denotes the set of all subsets of X. The *reachable map* of the polysystem $\{\pi_i | \, i \in I\}$ is the multivalued

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map $R : X \times \mathbb{R}^+ \to 2^X$ defined by

$$R(x,t) = \{y \in X | \text{ for some integer } n \geq 1, \text{ there exist } t_1, \cdots, t_n \in \mathbb{R}^+ \text{ and } i_1, \cdots, i_n \in I \text{ such that } \sum_{i=1}^{n} t_i = t \text{ and } y = \pi_{i_n}(\pi_{i_{n-1}}(\cdots \pi_{i_2}(\pi_{i_1}(x,t_1),t_2),\cdots,t_{n-1}),t_n)\}$$

Also, we define $R(A,t) = \bigcup_{x\in A} R(x,t)$ for $A \subset X$ and $t \in \mathbb{R}^+$ and we define $R(x,\mathbb{R}^+)$ by $R(x)$. For $A \subset X$, we let $R(A) = \bigcup_{x\in A} R(x)$. The multivalued map $DR : X \times \mathbb{R}^+ \to 2^X$ is defined by

$$DR(x,t) = \{y \in X | \text{ there exist sequences } x_n \to x, \ y_n \to y, \ t_n \to t \text{ such that } y_n \in R(x_n,t_n)\}.$$

Given $x \in X$, a subset M of X and $\delta > 0$, we denote

$$d(x,M) = \inf\{d(x,y) | \ y \in M\},$$
$$B(M,\delta) = \{x \in X | \ d(x,M) < \delta\}.$$

We say that a compact subset M of X is *stable* if, for any neighborhood U of M, there exists a neighborhood V of M such that $R(V) \subset U$; *a uniform attractor* if $A_U(M)$ contains a neighborhood of M, where

$$A_U(M) = \{x \in X | \text{ there exist a neighborhood } V \text{ of } x \text{ and } t \in \mathbb{R}^+ \text{ such that } R(V,[t,\infty)) \subset U \text{ for any neighborhood } U \text{ of } M\};$$

an attractor if $A(M)$ contains a neighborhood of M, where

$$A(M) = \{x \in X | \text{ there exists } t \in \mathbb{R}^+ \text{ such that } R(x,[t,\infty)) \subset U \text{ for any neighborhood } U \text{ of } M\};$$

and *asymptotically stable* if M is stable and an attractor.

Given any compact subset M of X and positively invariant neighborhood W of M, a *Lyapunov function* of M is a continuous function $\phi : W \to \mathbb{R}^+$ such that

(1) $\phi(x) = 0$ if and only if $x \in M$, and
(2) for each $y \in R(x,t), \phi(y) \leq \phi(x)$.

We conclude this section with the results from Gu and Ryu [2, 3] which are necessary in the following section.

Proposition 1.1 (Gu and Ryu [2]). *A compact subset M of X is stable if and only if, for any neighborhood U of M, there is a compact positively invariant neighborhood V of M such that $V \subset U$.*

Proposition 1.2 (Gu and Ryu [2]). *Let M be a compact subset of X and suppose M is stable. Then there is a neighborhood W of M such that a cluster map R is uniformly bounded on W.*

Proposition 1.3 (Gu and Ryu [2]). *Let a compact subset M of X be asymptotically stable. Then M is a uniform attractor.*

Proposition 1.4 (Gu and Ryu [3]). *Let a compact subset M of X be stable. Then there is a neighborhood W of M such that a function l is continuous in $W \times \mathbb{R}^+$, where a function $l : X \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $l(x,t) = \sup_{y \in R(x,t)} d(y, M)$.*

Theorm 1.5 (Gu and Ryu [3]). *Suppose that a compact subset M of X is asymptotically stable. Let a real valued function ϕ be defined by*

$$\phi(x) = \sup_{t \in \mathbb{R}^+} \frac{l(x,t)}{1 + l(x,t)}.$$

Then a function ϕ is a Lyapunov function of M.

2. A strict Lyapunov function

In this section, we shall show that asymptotic stability is characterized in terms of a strict Lyapunov function.

Definition 2.1. Let M be a compact subset of X and W a positively invariant neighborhood of M. A *strict Lyapunov function* of M is a Lyapunov function ϕ of M defined on W that satisfies the following properties:

(1) $\phi(y) < \phi(x)$ for any $y \in R(x,t), x \notin M$ and $t > 0$, and

(2) $\phi(y) = \phi(z)$ for $y, z \in \Lambda(x)$, where $\Lambda(x)$ is the limit set of x in X.

Theorm 2.2. *Let M be a compact subset of X. Then M is asymptotically stable if and only if a strict Lyapunov function of M exists.*

Proof. (Necessity) Let M be asymptotically stable. From Propositions 1.1–1.4 and Theorem 1.5, there is a positively invariant neighborhood W of M such that $W \subset A_U(M)$, a cluster map R is uniformly bounded on W, l is continuous in $W \times \mathbb{R}^+$ and $\phi : W \to \mathbb{R}^+$ is Lyapunov function of M.

We define a function $f : W \times \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f(x,t) = \sup_{y \in R(x,t)} \phi(y), \text{ where } \phi(y) = \sup_{t \in \mathbb{R}^+} \frac{l(y,t)}{1 + l(y,t)}.$$

Then this function is well defined on $W \times \mathbb{R}^+$ and satisfies the properties :

(1) $f(x,t)$ is continuous in $W \times \mathbb{R}^+$,
(2) $f(x,t) = 0$ for each $x \in M, t \in \mathbb{R}^+$,
(3) $f(x,0) > 0$ for each $x \in W - M$,
(4) $f(x,t) \leq f(x,t+s)$ for each $y \in R(x,s), t \in \mathbb{R}^+$,
(5) $f(x,t) \leq f(x,s)$ for all $t \geq s, x \in W$, and
(6) $f(x,t) \to 0$ as $t \to \infty$.

First, we show that $f(x,t)$ is continuous in $W \times \mathbb{R}^+$. Assume that $f(x,t)$ is not continuous at $(x,t) \in W \times \mathbb{R}^+$. Then there is a sequence $(x_n, t_n) \in W \times \mathbb{R}^+$ with $(x_n, t_n) \to (x,t)$ such that $f(x_n, t_n) \not\to f(x,t)$. Since $0 \leq f(x_n, t_n) \leq 1$, we have $f(x_n, t_n) \to \mu$, $\mu \neq f(x,t)$.

Let $\mu < f(x,t)$. For each $y \in R(x,t)$, since R is upper semicontinuous at (x,t), there is a sequence $y_n \in R(x_n, t_n)$ such that $y_n \to y$. Clearly, $f(x_n, t_n) \geq \phi(y_n)$. Thus we have

$$\mu = \lim_{n \to \infty} f(x_n, t_n) \geq \lim_{n \to \infty} \phi(y_n) = \phi(\lim_{n \to \infty} y_n) = \phi(y).$$

It follows that $\mu \geq f(x,t)$. This contradicts the fact that $\mu < f(x,t)$. Next, let $\mu > f(x,t)$. Choose α with $\mu > \alpha > f(x,t)$. We may assume that $f(x_n, t_n) > \alpha$. For each integer n, there is a sequence $y_n \in R(x_n, t_n)$ such that $\phi(y_n) > \alpha$. Since R is uniformly bounded on W, there is a neighborhood U of x such that $\overline{R(U, \mathbb{R}^+)}$ is compact.

We may assume that $x_n \in U$. Since $y_n \in R(x_n, t_n) \subset R(U, \mathbb{R}^+)$ and $\overline{R(U, \mathbb{R}^+)}$ is compact, we have $y_n \to y$. Thus we obtain $y \in DR(x,t)$. Since R is a cluster map, we have $y \in R(x,t)$ and so $f(x,t) \geq \phi(y)$. Since $\phi(y_n) > \alpha$, we have

$$\phi(y) = \phi(\lim_{n \to \infty} y_n) = \lim_{n \to \infty} \phi(y_n) \geq \alpha > f(x,t).$$

This is a contradiction. Hence f is continuous in $W \times \mathbb{R}^+$.

In order to prove (2), let $x \in M$. Then it is clear that $\phi(x) = 0$. For each $y \in R(x,t)$, since $\phi(y) \leq \phi(x)$, we have $\phi(y) = 0$. Thus it follows that $f(x,t) = 0$.

Next, let $x \in W - M$. Then we have $\phi(x) > 0$. Since $x \in R(x,0)$, we have $f(y,t) \geq \phi(x) > 0$. Therefore, (3) is proved. We will show that $f(y,t) \leq f(x,t+s)$. For each $y \in R(x,s), t \in \mathbb{R}^+$, we have

$$f(x,t) = \sup_{z \in R(y,t)} \phi(z) \leq \sup_{z \in R(R(x,s),t)} \phi(z) = \sup_{z \in R(x,s+t)} \phi(z) = f(x,s+t).$$

To show that $f(x,t) \leq f(x,s)$. For all $t \geq s, x \in W$,

$$f(x,t) = \sup_{y \in R(x,t)} \phi(y) = \sup_{y \in R(R(x,s),t-s)} \phi(y).$$

For each $y \in R(R(x,s),t-s)$, there is $z \in R(x,s)$ such that $y \in R(z,t-s)$. Thus we have $\phi(y) \leq \phi(z) \leq f(x,s)$ and so $f(x,t) \leq f(x,s)$. Thus (5) is completed.

Finally, we show (6). For each $\varepsilon > 0$, let $U = \{x \in W | \, \phi(x) < \varepsilon\}$ be a neighborhood of M. Since $x \in A(M)$, there is a $s \in \mathbb{R}^+$ such that $R(x,[s,\infty)) \subset U$. For any $y \in R(x,t)$, $t \geq s$, we have $R(x,t) \subset R(x,[s,\infty)) \subset U$. Thus $y \in U$ and so $\phi(y) < \varepsilon$. We have $f(x,t) \leq \varepsilon$. Since ε is arbitrary, we have $f(x,t) \to 0$ as $t \to \infty$. Hence (6) is proved.

We construct a function $\Psi : W \to \mathbb{R}^+$ by

$$\Psi(x) = \int_0^\infty e^{-t} f(x,t)dt.$$

Then this function Ψ is well defined and is continuous on W. Let $x \in M$. For any $t \in \mathbb{R}^+$, we have $f(x,t) = 0$. Thus $\Psi(x) = 0$. Let $x \in W - M$. Then $f(x,0) > 0$. It follows that $\Psi(x) > 0$. Let $y \in R(x,s)$. Then

$$\Psi(y) = \int_0^\infty e^{-t} f(y,t)dt \leq \int_0^\infty e^{-t} f(x,s+t)dt \leq \int_0^\infty e^{-t} f(x,t)dt = \Psi(x).$$

Let $x \in W - M$, $y \in R(x,s)$ and $s > 0$. We claim that $\Psi(y) < \Psi(x)$. Suppose that $\Psi(y) = \Psi(x)$. Then, for each $t \in \mathbb{R}^+$, we have $f(x,s+t) = f(x,t)$. Let $t = ns$, $n = 0,1,\cdots$. For each n, we have $f(x,0) = f(x,ns)$. Now, $\lim_{n\to\infty} f(x,ns) = 0$ and so $f(x,0) = 0$. This contradiction shows that $\Psi(y) < \Psi(x)$. Let $x \in W$. Then we have $x \in A(M)$. Thus $\Lambda(x) \neq \emptyset \subset M$. For each $y \in \Lambda(x)$, we have $R(y,\mathbb{R}^+) \subset M$. For any $t \in \mathbb{R}^+, f(y,t) = 0$. It follows that $\psi(y) = 0$.

Hence Ψ is a strict Lyapunov function of M.

(Sufficiency) Let $\phi : W \to \mathbb{R}^+$ be a strict Lyapunov function of M. Assume that there is a neighborhood U of M such that for any neighborhood V of M,

$R(V,\mathbb{R}^+) \not\subset U$. We can choose a $\varepsilon > 0$ so that $\overline{B(M,\varepsilon)} \subset U$ and $\overline{B(M,\varepsilon)}$ is compact. For each integer n, we have

$$R(B(M,\frac{\varepsilon}{n}),\mathbb{R}^+) \not\subset \overline{B(M,\varepsilon)}.$$

Thus there is a sequence $x_n \in B(M,\frac{\varepsilon}{n})$ such that $R(x_n) \not\subset \overline{B(M,\varepsilon)}$. Since R is $c-c$ map, we have $R(x_n) \cap \partial\overline{B(M,\varepsilon)} \neq \emptyset$. We can choose a sequence $y_n \in R(x_n) \cap \partial\overline{B(M,\varepsilon)}$. Since $\partial\overline{B(M,\varepsilon)}$ is compact, we have $y_n \to y \in \partial\overline{B(M,\varepsilon)}$. Clearly, $x_n \to x \in M$. Since $\phi(y_n) \leq \phi(x_n)$,

$$\phi(y) = \phi(\lim_{n\to\infty} y_n) = \lim_{n\to\infty} \phi(y_n) \leq \lim_{n\to\infty} \phi(x_n) = \phi(\lim_{n\to\infty} x_n) = \phi(x) = 0.$$

We have $\phi(y) = 0$ and so $y \in M$. This is a contradiction. Hence, for any neighborhood U of M, there is a neighborhood V of M such that $R(V,\mathbb{R}^+) \subset U$. It follows that M is stable.

By Proposition 1.1, there is a compact positively invariant U of M such that $U \subset W$. For each $x \in U$, we have $\overline{R(x,\mathbb{R}^+)} \subset U$. Thus $\overline{R(x,\mathbb{R}^+)}$ is compact and so $\Lambda(x) \neq \emptyset$. Assume that $\Lambda(x) \not\subset M$. Then there is $y \in \Lambda(x)$ such that $y \notin M$. Take $z \in R(y,t)$, $t > 0$. Then we have $z \in R(y,t) \subset R(\Lambda(x),t) \subset \Lambda(x)$. Thus we have $\phi(z) = \phi(y)$. This contradicts the fact that $\phi(z) < \phi(y)$. Hence $\Lambda(x) \subset M$ and so $x \in A(M)$. Clearly, $U \subset A(M)$.

It follows that $A(M)$ is a neighborhood of M and so M is an attractor. Therefore M is asymptotically stable. Hence the proof of the theorem is complete. □

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