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REPRESENTATION OF SOLUTIONS OF FREDHOLM EQUATIONS IN $W_2^2(\Omega)$ OF REPRODUCING KERNELS

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ABSTRACT. In this paper we derive a decomposition of the solution of Fredholm equations of the second kind in terms of reproducing kernels in the space $W_2^1(\Omega)$.

1. INTRODUCTION

The solution problem in integral equations is to find an effective decomposition of the solution u of the form $u = \sum c_n s_n$, where the coefficients c_n and harmonics s_n have computable and relevant properties, optimal and desired duration properties in terms of the information area, respectively. (*e. g.*, Benedetto & Walnut [2]).

Let $\phi(x, y)$ be bounded on $\Omega = [a, b] \times [c, d]$ in $\mathbb{R}^2$ and let

$$W_2^2(\Omega) = \left\{u(x,y) \,\middle|\, u(x,y) \text{ is absolutely continuous and } \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x \partial y} \in L^2(\Omega)\right\}.$$

It is evident from the work of Cui, Deng & Wu [4], Cui, Lee & Lee [5] that the decomposition of the solution of Fredholm equations depends on $W_2^1(\mathbb{R})$, the space of reproducing kernels.

The present work addresses the solution representation based on Cui's approach by replacing $\mathbb{R}$ into Ω and we formulate the structure of this decomposition in terms of the reproducing kernels in $W_2^2(\Omega)$.

2. SPACE $W_2^2(\Omega)$ AND REPRODUCING KERNELS

Throughout, $L^2(\Omega)$ denotes, as usual, the Hilbert space of all Lebesgue square integrable functions on Ω and $R(A)$ the range of bounded linear operator A on

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$W_2^2(\Omega)$. The needed facts about reproducing kernels can be found in Lee, Lee & Kim [7], Saitoh [9].

Definition 2.1. Let $\phi(x,y)$ be bounded on $\Omega = [a,b] \times [c,d]$. If

$$\phi(I_i) = b_i d_i + a_i c_i - a_i d_i - b_i c_i \geq 0 \tag{1}$$

for $I_i = [a_i, b_i) \times [c_i, d_i) \subset \Omega$, where $[a_i, b_i)$ and $[c_i, d_i)$ are finite pairwise disjoint subintervals of $[a,b]$ and $[c,d]$, respectively. Then ϕ is called 'monotone' on Ω.

Definition 2.2. Let $\phi(x,y)$ be bounded and monotone on $\Omega = [a,b] \times [c,d]$. If, given any $\epsilon > 0$, there exists $\delta > 0$ such that $\min\{|b_i - a_i|, |d_i - c_i|\} < \delta$ implies $\sum_1^n \phi(I_i) < \epsilon$, then ϕ is called 'absolutely continuous' on Ω.

We now define, for $u, v \in W_2^2(\Omega)$,

$$\langle u, v\rangle = \int\int_{\Omega} \Big(u(x,y)v(x,y) + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y}\Big) d\sigma \tag{2}$$

as inner product on $W_2^2(\Omega)$ and norm $||u||_{W_2^2} =< u, u >^{\frac{1}{2}}$.

Theorem 2.3. *Let $f \in W_2^2(\Omega)$. Then $||f||_\infty = \sup_{p\in\Omega}|f(p)| \leq M||f||_{W_2^2}$, where*

$$M = \frac{1}{\sqrt{(b-a)(d-c)}}\big[(b-a)(d-c) + (b-a) + (d-c) + 1\big].$$

Proof. Let $(x_1, y_1), (x_2, y_2) \in \Omega$. Then

$$\begin{aligned} f(x_2, y_2) = &\int_{x_1}^{x_2}\int_{y_1}^{y_2} f''_{t\tau} d\tau dt + \int_{x_1}^{x_2} f'_t(t, y_1) dt \\ &+ \int_{y_1}^{y_2} f'_\tau(x_1, \tau) d\tau + f(x_1, y_1). \end{aligned} \tag{3}$$

Let the both sides of (3) integrate over Ω. Then, Cauchy-Schwartz, and Hölder's inequalities allow us to compute

$$\begin{aligned} \int_\Omega |f(x_2,y_2)| dx_1 dy_1 = \int_\Omega \Big| &\int_{x_1}^{x_2}\int_{y_1}^{y_2} f''_{t\tau}(t,\tau) d\tau dt + \int_{x_1}^{x_2} f'_t(t, y_1) dt \\ &+ \int_{y_1}^{y_2} f'_\tau(x_1, \tau) d\tau + f(x_1, y_1)\Big| dx_1 dx_2. \end{aligned} \tag{4}$$

On the other hand, since

$$\begin{aligned}
&|f(x_2,y_2)|(b-a)(d-c)\\
&\le (b-a)(d-c)\int_a^b\int_c^d\Big|f''_{t\tau}(t,\tau)\Big|dtd\tau+(b-a)\int_c^d\int_a^b\Big|f'_t(t,y_1)\Big|dtdy_1\\
&\quad+(d-c)\int_a^b\int_c^d\Big|f'_\tau(x_1,\tau)\Big|d\tau dt+\int_a^b\int_c^d\Big|f(x_1,y_1)\Big|dx_1dy_1\\
&\le (b-a)(d-c)\Big\{(b-a)(d-c)\Big(\int_a^b\int_c^d\Big|f''_{t\tau}(t,\tau)\Big|^2d\tau dt\Big)^{\frac12}\\
&\quad+(b-a)\Big(\int_c^d\int_a^b\Big|f'_t(t,y_1)\Big|^2dtdy_1\Big)^{\frac12}+(d-c)\Big(\int_a^b\int_c^d\Big|f'_\tau(x_1,\tau)\Big|^2d\tau dx_1\Big)^{\frac12}\\
&\quad+\Big(\int_a^b\int_c^d\Big|f(x_1,y_1)\Big|^2dx_1dy_1\Big)^{\frac12}\Big\},
\end{aligned}$$

we have

$$\begin{aligned}
|f(x_2,y_2)|&\le\Big[\frac{(b-a)(d-c)+(b-a)+(d-c)+1}{(b-a)(d-c)}\Big]^{\frac12}\\
&\times\Big\{\int_a^b\int_c^d\ \Big\{|f''_{t\tau}(t,\tau)|^2+|f'_t(t,y_1)|^2+|f'_\tau(x_1,\tau)|^2+|f''_{t\tau}(t,\tau)|^2\Big\}dx_1dy_1\Big\}^{\frac12}\\
&=\quad M||f||_{W_2^2}.
\end{aligned}$$

□

We recall, (see Cui & Deng [3], Cui, Deng & Wu [4]), that any $u(x)\in W_2^1([a,b])$ can be represented in terms of a reproducing kernel such as $u(x)=\langle u(\cdot),R_x(\cdot)\rangle$, where

$$R_x(y)=\frac{1}{2sh(b-a)}\big[ch(y+x-a-b)+ch(|y-x|+a-b)\big],y\in[a,b].$$

Our next result extends this formula to the case of $\Omega=[a,b]\times[c,d]$.

Theorem 2.4. *$R_{xy}(\cdot,\cdot)$ is a reproducing kernel of space $W_2^2(\Omega)$, where $R_{xy}(\cdot,\cdot)=R_x(\cdot)R_y(\cdot)$.*

Proof. For any $u\in W_2^2(\Omega)$,

$$\begin{aligned}
\langle u(\cdot,\cdot),R_{xy}(\cdot,\cdot)\rangle&=\int_c^d d\eta\int_c^d\Big\{u(\xi,\eta)R_{xy}(\xi,\eta)+u'_\xi(\xi,\eta)R'_{xy}(\xi,\eta)_\xi\\
&\qquad+u'_\eta(\xi,\eta)R'_{xy}(\xi,\eta)_\eta+u''_{\xi\eta}(\xi,\eta)R''_{xy}(\xi,\eta)_{\xi\eta}\Big\}d\xi\\
&=\int_c^d\Big\{\int_a^b u(\xi,\eta)R_{xy}(\xi,\eta)d\xi+u'_\xi(\xi,\eta)R'_{xy}(\xi,\eta)_\xi
\end{aligned}$$

$$+ u'_\eta(\xi,\eta)R'_{xy}(\xi,\eta)_\eta + u''_{\xi\eta}(\xi,\eta)R''_{xy}(\xi,\eta)_{\xi\eta}\Big\}d\xi$$
$$= \int_c^d d\eta\Big\{R_y(\eta)\int_a^b \big(u(\xi,\eta)R_x(\xi) + u'_\xi(\xi,\eta)R'_x(\xi)_\xi\big)d\xi$$
$$+ R'_y(\eta)_\eta \int_a^b \big(u'_\eta(\xi,\eta)R_x(\xi) + u''_{\xi\eta}R'_x(\xi)_\xi\big)d\xi\Big\}.$$

Applying the reproducing properties of $R_y(\xi)$, we have

$$\int_a^b \Big[u(\xi,\eta)R_x(\xi) + u'_\xi(\xi,\eta)R'_x(\xi)_\xi\Big]d\xi = \langle u(\cdot,\eta), R_x(\cdot)\rangle = u(x,\eta).$$

$$\int_a^b u'_\eta(\xi,\eta)R_x(\xi) + \frac{\partial}{\partial\xi}(u'_\eta(\xi,\eta))R'_x(\xi)_\xi d\xi = \langle u'_\eta(\cdot,\eta), R'_x(\cdot)\rangle = u'_\eta(x,\eta).$$

So that,

$$\langle u(\cdot,\cdot), R_{xy}(\cdot,\cdot)\rangle = \int_c^d \Big\{ u(x,\eta)R_y(\eta) + u'_\eta(x,\eta)R'_x(\eta) \Big\}d\eta.$$

Finally, we use the definition of reproducing kernels to get the conclusion. □

3. DECOMPOSITION BY REPRODUCING KERNELS

Let us consider the following classical Fredholm equation

$$u(x,y) - \lambda\int_a^b\int_c^d k(x,y;t,\tau)u(t,\tau)dtd\tau = f(x,y),$$

and let

$$(I - \lambda\mathbb{K})u = f, \tag{5}$$

where I is identity operator, λ is parameter, and

$$\mathbb{K}u = \int_a^b\int_c^d k(x,y;t,\tau)u(t,\tau)dtd\tau.$$

Then, for a given sequence (p_i) in $\Omega = [a,b]\times[c,d]$, an elementary linear algebra argument shows that $(\phi_j(p_i))$ is linearly independent Lee, Lee, & Kim [7], where $\phi_j(p_i) = R_{p_j}(p_i)$ for $p_i \in \Omega$.

We begin by stating the next results, whose conclusion will be needed for our purposes.

Lemma 3.1. *Let A be a bounded linear operator on $W_2^2(\Omega)$, A^* be adjoint of A, and (p_i) be dense in Ω. Then $(A^*\phi_j(p_i))$ is complete if and only if A is one-to-one.*

Proof. Assume A is one-to-one and for $u \in W_2^1(\Omega)$, let $\langle u, A^*\phi_j(p_i)\rangle = 0$. Then $\langle u, A^*\phi_j(p_i)\rangle = \langle Au, \phi_j(p_i)\rangle = (Au)(p_i) = 0$. Thus, the assumption of A implies $u = 0$. Conversely, let $Au = 0$. For each $\phi_j(p_i)$, $\langle Au, \phi_j(p_i)\rangle = \langle u, A^*\phi_j(p_i)\rangle = 0$. Hence we have $u = 0$. □

Lemma 3.2. *If $\overline{R(A)} = W_2^2(\Omega)$, then $(A^*\phi_j(p_i))$ is linearly independent.*

Proof. Let $c = (c_i) \in \mathbb{C}$, and let $\sum c_j A^*\phi_j(p_i) = 0$. Then, for

$$u \in W_2^2(\Omega),\ \langle Au, \sum c_j\phi_j\rangle = 0.$$

It follows easily that $\sum c_j\phi_j(p_i) \in \overline{R(A)}^{\perp} = \{o\}$. So that we have $c_j = 0$. □

We use now Lemmas 3.1 and 3.2 to prove the following characterization of reproducing kernels.

Theorem 3.3. *Let $Au = f$ from (5), where $A = I - \lambda\mathbb{K}$ and $p = (p_i)$ be dense in Ω. Then u is decomposed by reproducing kernels.*

Proof. The above Lemmas show that $(A^*\phi_j(p))$ is complete and linearly independent. We now let $\psi_j(p) = (A^*\phi_j(p))$. By using Gram-Schmidt orthogonalization procedure of (ψ_j), we obtain $(\tilde{\psi_j})$ such that, as usual, $\langle\tilde{\psi_i}, \tilde{\psi_j}\rangle = 1$ if $i = j$ and 0 if $i \neq j$. Let $\tilde{\psi_i}(p) = \sum_{j=1}^{i} \beta_{ij}\psi_j(p)$. Then, we have

$$\begin{aligned} u &= \sum_{k=1}^{\infty}\langle u, \tilde{\psi_k}\rangle\tilde{\psi_k} = \sum_{k=1}^{\infty}\langle u, A^*\sum_{j=1}^{k}\beta_{kj}\phi_j(p))\rangle\tilde{\psi_k} \\ &= \sum_{k=1}^{\infty}\sum_{j=1}^{k}\beta_{kj}\langle Au, \phi_j\rangle\tilde{\psi_k}(p) = \sum_{k=1}^{\infty}\sum_{j=1}^{k}\beta_{kj}f(p)\tilde{\psi_k}(p) \\ &= \sum_{k=1}^{\infty}\tilde{f_k}\tilde{\psi_k}(p), \end{aligned}$$

where $\tilde{f_k} = \sum_{j=1}^{k}\beta_{kj}f(p)$, and the theorem is proved. □

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