J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 9, Number 1 (May 2002), Pages 47–55

MINTY'S LEMMA FOR (θ, η)-PSEUDOMONOTONE-TYPE SET-VALUED MAPPINGS AND APPLICATIONS

BYUNG-SOO LEE AND JAE-DUK NOH

ABSTRACT. In this paper, we consider a Minty's lemma for (θ, η)-pseudomonotone-type set-valued mappings in real Banach spaces and then we show the existence of solutions to variational-type inequality problems for (θ, η)-pseudomonotone-type set-valued mappings in nonreflexive Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

Minty [14] showed the linearization lemma for the scalar case, which has played useful roles in variational inequalities. In fact, the classical Minty's inequality and Minty's lemma have been shown to be important tools in the regularity results of the solution for a generalized non-homogeneous boundary value problem (*cf.* Baiocchi & Capelo [2]) and, when the operator is a gradient, also a minimum principle for convex optimization problems (*cf.* Kinderlehrer & Stampacchia [8]). And Behera & Panda [3] obtained a nonlinear generalization of Minty's lemma. Furthermore they applied the result to obtain a solution of a certain variational-like inequality. Kassay & Kolumban [7] considered the following Minty-type problem for set valued mappings with two variables for the scalar case: Find an element $x \in K$ such that

$$\sup_{z \in T(x,x)} \langle z, y - x \rangle \geq 0 \quad \text{for} \quad y \in K,$$

where K is a nonempty convex subset of a dual space X^* of a Banach space X and $T : K \times K \to 2^X$ is a set-valued mapping.

In 1995, Chang, Lee & Chen [5] firstly considered variational inequalities for monotone single-valued mappings in nonreflexive Banach spaces. Later on, Verma [9] and Watson [16] studied variational inequalities for strong pseudomonotonicity

Received by the editors October 13, 2001 and, in revised form, April 17, 2002.
2000 *Mathematics Subject Classification.* 49J40.
Key words and phrases. variational-type inequalities, (θ, η)-pseudomonotone-type, Minty's lemma, nonreflexive Banach spaces, hemicontinuity.

and pseudomonotonicity for single-valued mappings in nonreflexive Banach spaces, respectively.

Recently, B.-S. Lee & G.-M. Lee [10] introduced (η, θ)-pseudomonotonicity, which generalizes and extends monotonicities mentioned in Chang, Lee & Chen [5]. and showed the existence theorem of solutions to generalized variational-like inequalities for single-valued mappings in nonreflexive Banach spaces which generalizes and extends some results in Chang, Lee & Chen [5], Verma [15] and Watson [16].

Let X be a real Banach space, $T : K \to 2^{X^*}$, where X^* is the dual space of X, a set-valued mapping, $\theta : K \times K \to X^*$ an operator and $\eta : K \times K \to \mathbb{R}$ a function. The scalar variational-type inequality problem for set-valued mappings is to find an $x_0 \in K$ such that for all $x \in K$ there exists $v_0 \in T(x_0)$ satisfying

$$\langle v_0, \theta(x, x_0) \rangle + \eta(x_0, x) \geq 0.$$

Behera and Panda [4] firstly considered variational-type inequalities for single-valued mapping. B.-S. Lee & S.-J. Lee [12, 13] extended to the vector case for single-valued mappings using generalized Minty's lemma and for set-valued mappings using Fan's geometrical lemma, respectively.

In this paper, we consider a Minty's lemma for (θ, η)-pseudomonotone-type set-valued mappings in real Banach spaces and then we show the existence of solutions to the variational-type inequality problems for (θ, η)-pseudomonotone-type set-valued mappings in nonreflexive Banach spaces.

The method of the proofs shown in Theorem 2.2 and Theorem 3.1 below is the same as the method in B.-S. Lee, G.-M. Lee & S.-J. Lee [11].

Definition 1.1. Let X be a real nonreflexive Banach space with the dual X^* and X^{**} be the dual of X^*. Let K be a subset of X^{**} and $T : K \to 2^{X^*} \setminus \{\varnothing\}$ a set-valued mapping, $\theta : K \times K \to X^{**}$ an operator and $\eta : K \times K \to \mathbb{R}$ a function.

(1) T is said to be *(θ, η)-pseudomonotone* if for all $x, y \in K$ and for all $u \in T(x)$, $v \in T(y)$, we have

$$\langle v, \theta(x, y) \rangle + \eta(y, x) \geq 0 \text{ implies } \langle u, \theta(x, y) \rangle + \eta(y, x) \geq 0.$$

(2) T is said to be *(θ, η)-pseudomonotone-type* if for all x, $y \in K$ and for all $v \in T(y)$, we have

$$\langle v, \theta(x, y) \rangle + \eta(y, x) \geq 0 \text{ implies } \langle u, \theta(x, y) \rangle + \eta(y, x) \geq 0$$

for some $u \in T(x)$.

Definition 1.2. Let $K \subset X^{**}$ and $\theta : K \times K \to X^{**}$ be a function. A set-valued mapping $T : K \to 2^{X^*}$ is said to be hemicontinuous, if for any $x, y \in K$ with $x + t(y - x) \in K$ for any $t \in [0,1]$, the multifunction

$$t \in [0,1] \longmapsto T(x + t(y-x)) \cdot \theta(y,x)$$

is upper semicontinuous (shortly, u.s.c.) at 0^+, where

$$T(x + t(y-x)) \cdot \theta(y,x) = \{\langle s, \theta(y,x)\rangle : s \in T(x + t(y-x))\}.$$

The set-valued mapping T is said to be finite-dimensional u.s.c. if for any finite-dimensional subspace F of X^{**} with $K_F = K \cap F \neq \varnothing$, $T : K_F \to 2^{X^*}$ is u.s.c. in the norm topology.

Lemma 1.1 (Aubin & Cellina [1]). *Let X, Y be topological vector spaces and $T : X \to 2^Y$ be a set-valued mapping.*

(1) *If K is a compact subset of X, T is u.s.c., and compact-valued, then $T(K)$ is compact.*
(2) *If T is u.s.c. and compact-valued, then T is closed.*

2. Minty type lemma

The following Minty's Lemma for monotone mappings and considered in Banach spaces (*cf.* Baiocchi & Capelo [2]).

Lemma 2.1 (Minty's Lemma). *Let K be a closed convex subset of a real Banach space X and X^* the dual of X. Let $T : K \to X^*$ be a hemicontinuous monotone mapping. Then the following are equivalent:*

(a) *There exists a $y_0 \in K$ such that*

$$\langle T(y_0), x - y_0\rangle \geq 0 \quad \textit{for all} \quad x \in K.$$

(b) *There exists a $y_0 \in K$ such that*

$$\langle T(x), x - y_0\rangle \geq 0 \quad \textit{for all} \quad x \in K.$$

In 1994, Yao [17] considered a generalization of Minty lemma for pseudomonotone mappings which is continuous on finite dimensional subspaces of a Banach space.

Now we consider some result with (θ,η)-pseudomonotone-type hemicontinuous set-valued mappings in real Banach spaces.

Theorem 2.2. *Let X be a real Banach space and K a nonempty convex subset of X^{**}. Let $\theta : K \times K \to X^{**}$ be an operator, $\eta : K \times K \to \mathbb{R}$ a function, $T : K \to 2^{X^*}$ an (θ, η)-pseudomonotone-type hemicontinuous set-valued mapping such that*

(i) *$\theta(x, x) = \bar{0}$ and $\eta(x, x) = 0$, for all $x \in K$,*
(ii) *for each $y \in K$, $x \mapsto \theta(x, y)$ is affine, and*
(iii) *for each $y \in K$, $x \mapsto \eta(y, x)$ is convex.*

Then the following variational-type inequality problems are equivalent:

(a) *There exists $x_0 \in K$ such that for all $x \in K$ there exists $v_0 \in T(x_0)$ satisfying*

$$\langle v_0, \theta(x, x_0) \rangle + \eta(x_0, x) \geq 0.$$

(b) *There exists $x_0 \in K$ such that for all $x \in K$ there exists $v \in T(x)$ satisfying*

$$\langle v, \theta(x, x_0) \rangle + \eta(x_0, x) \geq 0.$$

Proof. The statement (b) follows from (a) by the definition of (θ, η)-pseudomonotone-type of T. Conversely, suppose that (b) holds and set $x_t = x_0 + t(x - x_0), t \in (0, 1)$. Then $x_t \in K$ and tere exists $v_t \in T(x_t)$ such that

$$\langle v_t, \theta(x_t, x_0) \rangle + \eta(x_0, x_t) \geq 0.$$

By the conditions (ii) and (iii), we have

$$t \left\{ \langle v_t, \theta(x, x_0) \rangle + \eta(x_0, x) \right\} + (1 - t) \left\{ \langle v_t, \theta(x_0, x_0) \rangle + \eta(x_0, x_0) \right\} \geq 0$$

Consequently we have

$$\langle v_t, \theta(x, x_0) \rangle + \eta(x_0, x) \geq 0. \tag{2.1}$$

Suppose that (a) does not hold. Then there exists an $x \in K$ such that for all $v_0 \in T(x_0)$

$$\langle v_0, \theta(x, x_0) \rangle + \eta(x_0, x) < 0.$$

By hemicontinuity of T, there exists $t_0 > 0$ such that for any $t \in (0, t_0)$ and $s \in T(x_t)$,

$$\langle s, \theta(x, x_0) \rangle + \eta(x_0, x) < 0.$$

Hence we have for any $t \in (0, t_0)$ and $s \in T(x_t)$

$$\langle s, \theta(x, x_0) \rangle + \eta(x_0, x) < 0,$$

which contradicts (2.1). So (a) holds. □

Considering $T : K \to X^*$ instead of $T : K \to 2^{X^*}$, $\theta(x, x_0) = x - x_0$ and a zero function η in Theorem 2.2, we can obtain in Watson [16, Lemma 1] as a corollary, which generalizes in Chang, Lee & Chen [5, Lemma 2.1].

Corollary 2.3 (Watson [16]). *Let $T : K \subset X^{**}$ be a pseudomonotone hemicontinuos operator with convex domain K, and let $x_0 \in K$ be given. Then*

$$\langle T(x_0), x - x_0 \rangle \geq 0, \quad \textit{for all} \ \ x \in K.$$

3. Applications

Let K be a subset of a topological vector space X. Then a mapping $T : K \to 2^X$ is called a *Knaster-Kuratowski-Mazurkiewicz* (in short, KKM) *mapping* if for each nonempty finite subset N of K, $\operatorname{co} N \subset T(N)$, where co denotes the convex hull and $T(N) = \bigcup\{T(x) : x \in N\}$ (*cf.* Knaster, Kuratowski & Mazurkiewicz [9]).

KKM Theorem (Fan [6]). *Let K be a nonempty subset of a Hausdorff topological vector space X. Let a set-valued mapping $T : K \to 2^X$ be a KKM mapping such that $T(x)$ is closed for all $x \in K$ and compact at some $x \in K$. Then*

$$\bigcap_{x \in K} T(x) \neq \varnothing.$$

Now, we consider the existence theorem of solutions to variational-type inequality problems for (θ, η)-pseudomonotone-type finite-dimensional u.s.c. compact set-valued mappings in nonreflexive Banach spaces.

Theorem 3.1. *Let X be a real nonreflexive Banach space and K a nonempty bounded closed convex subset of X^{**}. Let $\theta : K \times K \to X^{**}$ be an operator, $\eta : K \times K \to \mathbb{R}$ a function and $T : K \to 2^{X^*}$ an (θ, η)-pseudomonotone-type finite-dimensional u.s.c. compact set-valued mapping such that*

(i) *$\theta(x, x) = \bar{0}$ and $\eta(x, x) = 0$, for all $x \in K$,*
(ii) *for each $y \in K$, $x \mapsto \theta(x, y)$ is affine,*
(iii) *for each $y \in K$, $x \mapsto \eta(y, x)$ is convex, and*
(iv) *for each $x \in K$, $y \mapsto \theta(x, y)$ and $y \mapsto \eta(y, x)$ are continuous.*

Then there exists $x_0 \in K$ such that for all $x \in K$ there exists $v_0 \in T(x_0)$ satisfying

$$\langle v_0, \theta(x, x_0) \rangle + \eta(x_0, x) \geq 0.$$

Proof. For each finite-dimensional subspace F of X^{**} with $K_F = K \cap F \neq \varnothing$, we first consider the following variational-type inequality:

Find $x_0 \in K_F$ such that for all $x \in K_F$ there exists $v_0 \in T(x_0)$ such that

$$\langle v_0, \theta(x, x_0) \rangle + \eta(x_0, x) \geq 0. \tag{3.1}$$

Since K_F is a nonempty bounded closed convex set in a finite-dimensional space F and $T : K_F \to 2^{X^*}$ is u.s.c. compact-valued, (3.1) has a solution $x_0 \in K_F$. In fact, define a set-valued mapping $G : K_F \to 2^F$ by, for each $y \in K_F$,

$$G(y) := \{x \in K_F : \text{ there exists } v \in T(x) \text{ such that } \langle v, \theta(y,x)\rangle + \eta(x,y) \geq 0\}.$$

First, we prove that G is a KKM mapping. Suppose to the contrary that G is not a KKM mapping. Then there exists a finite set $\{x_1, x_2, \cdots, x_n\}$ in K_F, $\alpha_i \geq 0, i = 1, 2, \cdots, n$ and $x \in \mathrm{co}\{x_1, x_2, \cdots, x_n\}$ such that

$$\sum_{i=1}^{n} \alpha_i = 1 \text{ and } x = \sum_{i=1}^{n} \alpha_i x_i \notin \bigcup_{i=1}^{n} G(x_i).$$

So, by the conditions (ii), (iii) and definition of a set-valued mapping G, we have for all $v \in T(x)$,

$$\begin{aligned}
\langle v, \theta(x,x)\rangle + \eta(x,x) &= \left\langle v, \theta\left(\sum_{i=1}^{n} \alpha_i x_i, x\right)\right\rangle + \eta\left(x, \sum_{i=1}^{n} \alpha_i x_i\right) \\
&\leq \left\langle v, \sum_{i=1}^{n} \alpha_i \theta(x_i, x)\right\rangle + \sum_{i=1}^{n} \alpha_i \eta(x, x_i) \\
&= \sum_{i=1}^{n} \big(\langle v, \theta(x_i, x)\rangle + \eta(x, x_i) \big) \\
&< 0.
\end{aligned}$$

By the condition (i), this is impossible. Therefore G is a KKM mapping.

Next, we show that $G(y)$ is closed in F, for all $y \in K_F$. Let $\{x_n\}$ be a sequence in $G(y)$ converging to $x_0 \in F$. Then there exists $v_n \in T(x_n)$ for each n such that

$$\langle v_n, \theta(y, x_n)\rangle + \eta(x_n, y) \geq 0.$$

By (1) of Lemma 1.1, $T(K_F)$ is compact, there exists $v_0 \in T(K_F)$ such that $v_n \to v_0$. Since T is closed by (2) of Lemma 1.1, $v_0 \in T(x_0)$. And by the condition (iii) and the fact that $\{\|\theta(y, x_n)\|\}$ is bounded, we have

$$\begin{aligned}
&\big|\langle v_n, \theta(y,x_n)\rangle + \eta(x_n,y) - \{\langle v_0, \theta(y,x_0)\rangle + \eta(x_0,y)\}\big| \\
&\quad\leq \big|\langle v_n, \theta(y,x_n)\rangle - \langle v_0, \theta(y,x_0)\rangle\big| + \big|\eta(x_n,y) - \eta(x_0,y)\big| \\
&\quad\leq \big|\langle v_n - v_0, \theta(y,x_n)\rangle\big| + \big|\langle v_0, \theta(y,x_n) - \theta(y,x_0)\rangle\big| + \big|\eta(x_n,y) - \eta(x_0,y)\big| \\
&\quad\leq \|v_n - v_0\| \cdot \|\theta(y,x_n)\| + \|v_0\| \cdot \|\theta(y,x_n) - \theta(y,x_0)\| + |\eta(x_n,y) - \eta(x_0,y)| \\
&\quad\longrightarrow 0
\end{aligned}$$

as $n \to \infty$.

Consequently, there exists $v_0 \in T(x_0)$ such that $\langle v_0, \theta(y,x_0)\rangle + \eta(x_0,y) \geq 0$. Hence $x_0 \in G(y)$, and $G(y)$ is closed in F. Moreover, $G(y)$ is compact from the compactness of K_F. By KKM Theorem, $\bigcap_{x\in K_F} G(x)$ is nonempty.

Letting $x_0 \in \bigcap_{x\in K_F} G(x)$, for all $x \in K_F$ there exists $v_0 \in T(x_0)$ satisfying

$$\langle v_0, \theta(x,x_0)\rangle + \eta(x_0,x) \geq 0.$$

Since $T : K \to 2^{X^*}$ is hemicontinuous, $T : K_F \to 2^{X^*}$ is hemicontinuous.

On the other hand, since T is (θ,η)-pseudomonotone-type, by Theorem 2.2, there exists $v \in T(x)$ such that

$$\langle v, \theta(x,x_0)\rangle + \eta(x_0,x) \geq 0 \text{ for all } x \in K_F. \tag{3.2}$$

Let $\Im = \{F \subset X^{**} : \dim(F) < +\infty \text{ and } K_F \neq \varnothing\}$, and we associate each $F \in \Im$ with a set

$$W_F := \{x_0 \in K_F : \text{ there exists } v \in T(x) \text{ such that } \langle v, \theta(x,x_0)\rangle + \eta(x_0,x) \geq 0, \text{ for all } x \in K_F\}. \tag{3.3}$$

By (3.2), we know that W_F is nonempty. Since $W_F \subset K$ and K is weak*-closed, the weak*-closure $\overline{W}_F^*$ of W_F is contained in K. Next, for any n elements $F_1, F_2, \cdots, F_n$ in $\Im$, let F be the subspace spanned by the union $\bigcup_{i=1}^n F_i$. Then it is obvious that $\dim(F)$ is finite and K_F is nonempty, hence F lies in $\Im$. Moreover, W_F is nonempty and contained in W_{F_i}, for each $i = 1, 2, \cdots, n$. Therefore, we have

$$\bigcap_{i=1}^n \overline{W}_{F_i}^* \neq \varnothing.$$

This implies that $\{\overline{W}_F^* : F \in \Im\}$ has the finite intersection property. Since K is bounded, by Banach-Alaoglu theorem, K is weak*-compact and $\{\overline{W}_F^* : F \in \Im\}$ has the nonempty intersection, i.e.,

$$\bigcap_{F\in\Im} \overline{W}_F^* \neq \varnothing.$$

Take $x_0 \in \bigcap_{F\in\Im} \overline{W}_F^*$. Then for each $F \in \Im$, $x_0 \in K$.

Next, we prove that there exists $v \in T(x)$ such that

$$\langle v, \theta(x,x_0)\rangle + \eta(x_0,x) \geq 0, \text{ for all } x \in K.$$

For a given $x \in K$, take $F \in \Im$ such that $x \in F$. Since $x_0 \in \overline{W}_F^*$, there exists a net x_λ in W_F such that $x_\lambda \to x_0$ in the weak*-topology. Hence, by (3.3), there exists

$v \in T(x)$ such that

$$\langle v, \theta(x, x_\lambda) \rangle + \eta(x_\lambda, x) \geq 0, \text{ for all } x \in K.$$

Hence, by Theorem 2.2, we see that there exists $v_0 \in T(x_0)$ such that

$$\langle v_0, \theta(x, x_0) \rangle + \eta(x_0, x) \geq 0, \text{ for all } x \in K. \qquad \square$$

Considering $T : K \to X^*$ instead of $T : K \to 2^{X^*}$, $\theta(x, x_0) = x - x_0$ and a zero function η in Theorem 3.1, we obtain Watson [16, Theorem 2] as a corollary, which generalizes Chang, Lee & Lee [5, Theorem 2.2].

Corollary 3.2 (Watson [16]). *Let $T : K \subset X^{**} \to X^*$ be a pseudomonotone hemicontinuous operator with weak-star closed convex domain K. Further assume there exists a nonempty subset X_0 contained in a weak-star compact convex subset X_1 of K such that the set*

$$D = \{v \in K : \langle T(u), u - v \rangle \geq 0, \quad \textit{for all} \quad u \in X_0\}$$

is weak-star compact or empty.

Then there exists an $x_0 \in K$ such that $\langle T(x_0), x - x_0 \rangle \geq 0$, for all $x \in K$.

Remark 3.1. We can obtain the same results as Theorem 2.2 and Theorem 3.1 for (θ, η)-pseudomonotone set-valued mappings, which also generalize some results in Chang, Lee & Lee [5] and Watson [16].

REFERENCES

1. J. P. Aubin and A. Cellina: *Differential Inclusions.* Springer-Verlag, Berlin, 1984. MR **85j**:49010
2. C. Baiocchi and A. Capelo: *Variational and Quasivatiational Inequalities, Applications to Free Boundary Problems.* John Wiley and Sons, Inc., New York, 1984. MR **86e**:49018
3. A. Behera and G. K. Panda: A generalization of Minty's lemma. *Indian J. pure appl. Math.* **28** (1997), 897–903. MR **98j**:47152
4. A. Behera and G. K. Panda: Generalized variational-type inequality in Hausdorff topological vector spaces. *Indian J. pure appl. Math.* **28** (1997), no. 3, 343–349. MR **98c**:49020
5. S.-S. Chang, B.-S. Lee and Y.-Q. Chen: Variational inequalities for monotone operators in nonreflexive Banach spaces. *Appl. Math. Lett.* **8** (1995), no. 6, 29–34. MR **96j**:47063
6. K. Fan: A generalization of Tychonoff's fixed point theorem. *Math. Ann.* **142** (1960/1961), 305–310. MR **24**#A1120

7. G. Kassay and J. Kolumban: Variational inequalities given by semi-pseudomonotone mappings. *Nonlinear Anal. Forum* **5** (2000), 35–50. MR **2001m:**47137
8. D. Kinderlehrer and G. Stampacchia: *An introduction to variational inequalities and their applications.* Academic Press, Inc., New York, 1980. MR **81g:**49013
9. B. Knaster, B. Kuratowski and S. Mazurkiewicz: Ein beweis des fixpunkfsatzes für n-dimensionale simplex. *Fund. Math.* **14** (1929), 132–137.
10. B.-S. Lee and G.-M. Lee: Variational inequalities for (η, θ)-pseudomonotone operators in nonreflexive Banach spaces. *Appl. Math. Lett.* **12** (1999), no. 5, 13–17. CMP 1 750 132
11. B.-S. Lee, G.-M. Lee and S.-J. Lee: Variational-type inequalities for (η, θ, δ)-pseudomonotone-type set-valued mappings in nonreflexive Banach spaces. *Appl. Math. Lett.* **15** (2002), no. 1, 109–114. CMP 1 867 606
12. B.-S. Lee and S.-J. Lee: A vector extension to Behera and Panda's generalization of Minty's lemma. *Indian J. Pure Appl. Math.* **31** (2000), 1483–1489. MR **2001m:**90099
13. ______: Vector variational-type inequalities for set-valued mappings. *Appl. Math. Lett.* **13** (2000), no. 3, 57–62. MR **2000m:**47095
14. G. J. Minty: On the generalization of a direct method of calculus of variations. *Bull. Amer. Math. Soc.* **73** (1967), 315–321. MR **35**#3501
15. R. U. Verma: Variational inequalities involving strongly pseudomonotone hemicontinuous mappings in nonreflexive Banach spaces. *Appl. Math. Lett.* **11** (1998), no. 2, 41–43. CMP 1 609 673
16. P. J. Watson: Variational inequalities in nonreflexive Banach spaces. *Appl. Math. Lett.* **10** (1997), no. 2, 45–48. CMP 1 436 496
17. J. C. Yao: Variational inequalities with generalized monotone operators. *Math. Oper. Res.* **19** (1994), no. 3, 691–705. MR **95g:**49021

(B. S. LEE) DEPARTMENT OF MATHEMATICS, KYUNGSUNG UNIVERSITY, DAEYEON-DONG, NAM-GU, BUSAN 608-736, KOREA
Email address: `bslee@ks.ac.kr`

(J. D. NOH) DEPARTMENT OF MATHEMATICS, KYUNGSUNG UNIVERSITY, DAEYEON-DONG, NAM-GU, BUSAN 608-736, KOREA
Email address: `jaedug1111@hanmail.net`