J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
Volume 11, Number 1 (February 2004), Pages 15–22
ISSN 1226-0657

ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR DIFFERENCE EQUATION $x_{n+1} = \alpha + \beta x_{n-1}^p / x_n^p$

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ABSTRACT. In this paper, we investigate asymptotic stability, oscillation, asymptotic behavior and existence of the period-2 solutions for difference equation

$$x_{n+1} = \alpha + \beta x_{n-1}^p / x_n^p$$

where $\alpha \geq 0, \beta > 0, |p| \geq 1$, and the initial conditions x_{-1} and x_0 are arbitrary positive real numbers.

1. INTRODUCTION

Consider the following recursive equation

$$x_{n+1} = \alpha + \beta \frac{x_{n-1}^p}{x_n^p} \tag{1.1}$$

where $\alpha \geq 0, \beta > 0, |p| \geq 1$ and the initial conditions x_{-1} and x_0 are arbitrary positive real numbers.

Recently, there has been an increasing interest in the study of the recursive sequences Amleh, Grove, Georgiou & Ladas [1], Gibbons, Kulenovic & Ladas [2], Kocić, Ladas & Rodrigues [3] and Kosmala, Kulenovic, Ladas & Teixeira [4]. In this paper, we study asymptotic stability, oscillation, asymptotic behavior and existence of the period-2 solutions for the difference equations (1.1).

We need the following definitions.

Definition 1. The *equilibrium point* $\overline{x}$ of the equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots$$

Received by the editors April 4, 2003 and, in revised form, January 11, 2004.
2000 *Mathematics Subject Classification.* 39A11.
Key words and phrases. difference equation, stability, oscillation, period-2 solution.
Research are supported by the Natural Science Foundation of Hebei Province.

is the point that satisfies the condition:

$$\overline{x} = F(\overline{x}, \overline{x}, \ldots, \overline{x}).$$

Definition 2. A *positive semi-cycle* of $\{x_n\}$ of equation (1.1) consists of "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$ all greater than or equal to the $\overline{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either } \ l = -1 \ \text{ or } \ l > -1 \ \text{ and } \ x_{l-1} < \overline{x},$$

and

$$\text{either } \ m = \infty \ \text{ or } \ m < \infty \ \text{ and } \ x_{m+1} < \overline{x}.$$

A *negative semi-cycle* of $\{x_n\}$ of equation (1.1) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$ all less than the $\overline{x}$, with $l \geq -1$ and $m \leq \infty$ and such that

$$\text{either } \ l = -1 \ \text{ or } \ l > -1 \ \text{ and } \ x_{l-1} \geq \overline{x}$$

and

$$\text{either } \ m = \infty \ \text{ or } \ m < \infty \ \text{ and } \ x_{m+1} \geq \overline{x}.$$

Definition 3. A solution $\{x_n\}$ of equation (1.1) is called *oscillatory* if $x_n - \bar{x}$ is neither eventually positive nor eventually negative. Otherwise, it is called *nonoscillatory*.

2. Main Results

First, we discuss asymptotic stability for equation (1.1).

Theorem 1. *f we assume $p \geq 1$, then following statements are true:*

(1) *The equilibrium point $\overline{x} = \alpha + \beta$ of equation (1.1) is locally asymptotically stable if $\alpha > (2p-1)\beta$.*

(2) *The equilibrium point $\overline{x} = \alpha + \beta$ of equation (1.1) is unstable if $0 \leq \alpha < (2p-1)\beta$.*

Proof. The linearized equation of the equation (1.1) about the equilibrium point $\overline{x} = \alpha + \beta$ is

$$y_{n+1} + \frac{p\beta}{\alpha+\beta} y_n - \frac{p\beta}{\alpha+\beta} y_{n-1} = 0. \tag{2.1}$$

The characteristic equation is given by

$$f(\lambda) = \lambda^2 + \frac{p\beta}{\alpha+\beta}\lambda - \frac{p\beta}{\alpha+\beta} = 0. \tag{2.2}$$

So by Linearized Stability Theorem Gibbons, Kulenovic Ladas [2] and Jury Criterion of Asymptotically Stable Kocić, Ladas & Rodrigues [3] $\overline{x} = \alpha + \beta$ is locally asymptotically stable if

$$f(-1) > 0, \quad f(1) > 0, \quad f(0) < 0$$

i. e.,

$$\alpha > (2p - 1)\beta,$$

and the equilibrium point $\overline{x} = \alpha + \beta$ is unstable if $0 \leq \alpha < (2p - 1)\beta$.

This completes the proof. □

Remark. If $\beta = 1$, we have the same result as in Amleh, Grove, Georgiou & Ladas [1].

Corollary 2. *If we assume $p \leq -1$, then following statements are true:*

(1) *The equilibrium point $\overline{x} = \alpha + \beta$ of equation (1.1) is locally asymptotically stable if $\alpha > -(p + 1)\beta$.*

(2) *The equilibrium point $\overline{x} = \alpha + \beta$ of equation (1.1) is unstable if $0 \leq \alpha < -(p + 1)\beta$.*

The proof is the same method as in Theorem 1.

The following are some results of oscillation and asymptotic behavior for the equation (1.1).

Theorem 3. *Assume $p \geq 1$, and let $\{x_n\}$ be a positive solution of equation (1.1) which consists of at least two semi-cycles. Then $\{x_n\}$ is oscillatory. Moreover with the possible exception of the first semi-cycle, every semi-cycle has length 1 and every term of $\{x_n\}$ is strictly greater than α, and with the possible exception of the first two semi-cycles, no term of $\{x_n\}$ is ever equal to $\alpha + \beta$.*

Proof. Consider the following two cases.

Case 1. Let $x_{N-1} < \alpha + \beta \leq x_N$ for some $N \geq 0$.

Then

$$x_{N+1} = \alpha + \beta \frac{x_{N-1}^p}{x_N^p} < \alpha + \beta,$$

and

$$x_{N+2} = \alpha + \beta \frac{x_N^p}{x_{N+1}^p} > \alpha + \beta.$$

Thus

$$x_{N+1} < \alpha + \beta < x_{N+2}.$$

Case 2. Let $x_N < \alpha + \beta \le x_{N-1}$ for some $N \ge 0$.

Then

$$x_{N+1} = \alpha + \beta \frac{x_{N-1}^p}{x_N^p} > \alpha + \beta,$$

and

$$x_{N+2} = \alpha + \beta \frac{x_N^p}{x_{N+1}^p} < \alpha + \beta.$$

Thus

$$x_{N+2} < \alpha + \beta < x_{N+1}.$$

This completes the proof. □

Theorem 4. *Suppose $p = -1$, and let $\{x_n\}$ be a positive solution of equation (1.1). Then $\{x_n\}$ is oscillatory. Moreover, with the possible exception of the first semi-cycle, the length of every semi-cycle is equal to 2 or 3, and every term of $\{x_n\}$ is strictly greater than α.*

Proof. Case 1. Let $x_{N-1} < \alpha + \beta$ and $x_N \le \alpha + \beta$ for some $N \ge 0$.

Then

$$x_{N+1} = \alpha + \beta \frac{x_N}{x_{N-1}} \tag{2.3}$$

from the above equality, we have

$$\frac{x_{N+1}}{x_N} = \frac{\alpha}{x_N} + \frac{\beta}{x_{N-1}} > \frac{\alpha}{\alpha + \beta} + \frac{\beta}{\alpha + \beta} = 1.$$

So,

$$x_{N+1} > x_N,$$

and thus,

$$x_{N+2} = \alpha + \beta \frac{x_{N+1}}{x_N} > \alpha + \beta.$$

Case 2. Let $x_{N-1} > \alpha + \beta$ and $x_N \ge \alpha + \beta$ for some $N \ge 0$. Then

$$\frac{x_{N+1}}{x_N} = \frac{\alpha}{x_N} + \frac{\beta}{x_{N-1}} < \frac{\alpha}{\alpha + \beta} + \frac{\beta}{\alpha + \beta} = 1.$$

So,

$$x_{N+1} < x_N$$

and so,

$$x_{N+2} = \alpha + \beta \frac{x_{N+1}}{x_N} < \alpha + \beta.$$

Case 3. Let $x_{N-1} < \alpha+\beta$ and $x_N \geq \alpha+\beta$ for some $N \geq 0$. Then $x_{N+1} > \alpha+\beta$;
Case 4. Let $x_{N-1} > \alpha+\beta$ and $x_N \leq \alpha+\beta$ for some $N \geq 0$. Then $x_{N+1} < \alpha+\beta$.

This completes the proof. □

Theorem 5. *Let $p \geq 1$, $0 \leq \alpha < 1 \leq \beta$, and $\{x_n\}$ be a solution of equation (1.1) such that*

$$0 < x_{-1} \leq \beta^{\frac{1}{p}} \quad and \quad x_0 \geq \left(\frac{\beta^2}{1-\alpha}\right)^{\frac{1}{p}}.$$

Then the following statements are true:

(1) $\lim\limits_{n\to\infty} x_{2n} = \infty$.
(2) $\lim\limits_{n\to\infty} x_{2n+1} = \alpha$.

Proof. Since $0 \leq \alpha < \beta$, so $\beta^2 - \alpha^2 < \beta^2$, and thus $\frac{\beta^2}{\beta-\alpha} > \alpha+\beta$.

Then

$$x_0^p \geq \frac{\beta^2}{1-\alpha} \geq \frac{\beta^2}{\beta-\alpha} > \alpha+\beta,$$

and we have

$$x_1 = \alpha + \beta\frac{x_{-1}^p}{x_0^p} \leq \alpha + \beta\frac{\beta}{x_0^p} \leq 1,$$

and

$$x_1 = \alpha + \beta\frac{x_{-1}^p}{x_0^p} > \alpha.$$

Thus

$$x_1 \in (\alpha, 1].$$

Similarly, we have

$$\begin{aligned} x_2 &= \alpha + \beta\frac{x_0^p}{x_1^p} \geq \alpha + \beta x_0^p, \\ x_3 &= \alpha + \beta\frac{x_1^p}{x_2^p} \leq \alpha + \beta\frac{1}{(\alpha + x_0^p)^p} \\ &\leq \alpha + \beta\frac{1}{\alpha + x_0^p} \leq \alpha + \frac{\beta^2}{x_0^p} \leq 1. \end{aligned}$$

Thus

$$x_3 \in (\alpha, 1].$$

Also

$$x_4 = \alpha + \beta\frac{x_2^p}{x_3^p} \geq \alpha + \beta x_2^p \geq \alpha + \beta(\alpha + x_0^p)^p$$
$$\geq \alpha + \beta(\alpha + x_0^p) = (1+\beta)\alpha + \beta x_0^p.$$

Thus

$$x_4 \geq (1+\beta)\alpha + \beta x_0^p.$$

By induction, we have

$$x_{2n} \geq \alpha \sum_{i=0}^{n-1} \beta^i + \beta^{n-1} x_0^p$$

and

$$\alpha < x_{2n+1} \leq 1.$$

Thus

$$\lim_{n\to\infty} x_{2n} = \infty.$$

and

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} \left(\alpha + \beta \frac{x_{2n-1}^p}{x_{2n}^p}\right) = \alpha.$$

This completes the proof. □

Finally, we study the existence of the period-2 solutions for equation (1.1).

Theorem 6. *Let $p = 1, \alpha > 0$. The following statements are true.*

(1) *Equation (1.1) has solutions of prime period 2 if and only if $\alpha = \beta$.*

(2) *Assume that $\alpha = \beta$ and $\{x_n\}$ be a solution of equation (1.1). Then x_n is periodic with period 2 if and only if $x_{-1} > \alpha, x_0 = \frac{\alpha x_{-1}}{x_{-1}-\alpha}$.*

Proof. (i) Let $\{x_n\}$ be a periodic solution of (1.1) with period 2. Then

$$x_{-1} = \alpha + \beta \frac{x_{-1}}{x_0}, \quad x_0 = \alpha + \beta \frac{x_0}{x_{-1}}.$$

Since $\alpha > 0, \beta > 0$, from the above equality, it implies $x_{-1} - \alpha \neq 0$ and $x_{-1} - \beta \neq 0$. Thus,

$$x_0 = \frac{\beta x_{-1}}{x_{-1}-\alpha}, \quad x_0 = \frac{\alpha x_{-1}}{x_{-1}-\beta}.$$

We have,

$$\frac{\beta x_{-1}}{x_{-1}-\alpha} = \frac{\alpha x_{-1}}{x_{-1}-\beta}.$$

Therefore,

$$(\alpha - \beta)x_{-1} - (\alpha^2 - \beta^2) = 0.$$

If $\alpha \neq \beta$, then $x_{-1} = \alpha + \beta$, we have

$$x_0 = \alpha + \beta, \text{ and } x_n = \alpha + \beta,$$

which contradicts $\{x_n\}$ is periodic with period 2.

If $\alpha = \beta$, for any $x_{-1} > \alpha$, set

$$x_0 = \frac{\alpha x_{-1}}{x_{-1} - \alpha}.$$

Then

$$x_1 = \alpha + \alpha\frac{x_{-1}}{x_0} = \alpha + \alpha\frac{x_{-1}(x_{-1} - \alpha)}{\alpha x_{-1}} = x_{-1}.$$

Similarly, we have $x_2 = x_0$. So $\{x_n\}$ is periodic with period 2.

(ii). From the proof of (i), for any $x_{-1} > \alpha$, set $x_0 = \frac{\alpha x_{-1}}{x_{-1}-\alpha}$, the solution $\{x_n\}$ is periodic with period 2; contrarily, if $\{x_n\}$ is the solution periodic with period 2 of (1.1), we have $x_1 = \alpha + \alpha\frac{x_{-1}}{x_0} = x_{-1}$, so $x_{-1} > \alpha, x_0 = \frac{\alpha x_{-1}}{x_{-1}-\alpha}$.

This completes the proof. □

Theorem 7. *Suppose $p = -1$. Then for any $\alpha \geq 0$, $\beta > 0$ equation (1.1) has no solution of prime period 2.*

Proof. If not, let $\{x_n\}$ be a solution of (1.1) which is periodic with period 2. From equation (1.1),
we have

$$x_{-1} = \alpha + \beta\frac{x_0}{x_{-1}}, \quad x_0 = \alpha + \beta\frac{x_{-1}}{x_0}.$$

It is evident that $x_{-1} > \alpha, x_0 > \alpha$.

So

$$x_0 = \frac{1}{\beta}({x_{-1}}^2 - \alpha x_{-1}) \text{ and } x_{-1} = \frac{1}{\beta}({x_0}^2 - \alpha x_0)$$

and we have

$${x_{-1}}^4 - 2\alpha {x_{-1}}^3 + \alpha(\alpha - \beta){x_{-1}}^2 + \beta(\alpha^2 - \beta^2)x_{-1} = 0,$$

$$x_{-1}(x_{-1} - \alpha - \beta)\left({x_{-1}}^2 + (\beta - \alpha)x_{-1} + \beta(\beta - \alpha)\right) = 0.$$

We obtain

$$x_{-1} = \alpha + \beta,$$

or

$$f(x_{-1}) = {x_{-1}}^2 + (\beta - \alpha)x_{-1} + \beta(\beta - \alpha) = 0. \tag{2.4}$$

If $x_{-1} = \alpha + \beta$, then

$$x_0 = \frac{1}{\beta}\left((\alpha + \beta)^2 - \alpha(\alpha + \beta)\right) = \alpha + \beta.$$

It is easy to see $x_n \equiv \alpha + \beta$, which have a contradiction.

Therefore $f(x_{-1}) = 0$.

(i) When $\alpha = \beta$, from (2.4), we has $x_{-1} = 0$, which contradicts $x_{-1} > \alpha$.

(ii) When $\alpha < \beta$, $\Delta = -(\beta - \alpha)(\alpha + 3\beta) < 0$, equation (2.4) have no real roots.
(iii) When $\alpha > \beta$, $f(0) = \beta(\beta - \alpha) < 0$, $f(\alpha) = \beta^2 > 0$.

Then, since the roots of (2.4) satisfy $x_{-1} < \alpha$, we has a contradiction.

Therefore for any $\alpha \geq 0$, $\beta > 0$, equation (1.1) has no solution of prime period 2. This completes the proof. □

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