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SKEW-ADJOINT INTERPOLATION ON $Ax = y$ IN ALG$\mathcal{L}$

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ABSTRACT. Given vectors x and y in a Hilbert space, an interpolating operator is a bounded operator T such that $Tx = y$. In this paper the following is proved: Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$ and let P_x be the projection onto $sp(x)$. If $P_xE = EP_x$ for each $E \in \mathcal{L}$, then the following are equivalent.

(1) There exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$, $Af = 0$ for all f in $sp(x)^\perp$ and $A = -A^*$.

(2) $\sup\left\{\frac{\|E^\perp y\|}{\|E^\perp x\|} : E \in \mathcal{L}\right\} < \infty$, $y \in sp(x)$ and $< x, y >= - < y, x >$.

1. INTRODUCTION

Suppose that we are given a Hilbert space $\mathcal{H}$ and a weakly closed algebra $\mathcal{C}$ of operators acting on $\mathcal{H}$. An *interpolation question* for $\mathcal{C}$ asks for which x and y is there a bounded operator $A \in \mathcal{C}$ such that $Ax = y$. The n-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison [8]. The 'n-vector interpolation problem', asks for an operator A such that $Ax_i = y_i$ for fixed finite collections $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$. In case $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance [9]: his result was extended by Hopenwasser [4] to the case that $\mathcal{U}$ is a CSL-algebra. Munch [10] obtained conditions for interpolation in case A is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser [5] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contains a sufficient condition for interpolation n-vectors. We Jo & Kang [6] obtained conditions for interpolation in the case A is in Alg$\mathcal{L}$ when $\mathcal{L}$ is a CSL.

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Again we Jo & Kang [7] showed an interpolation condition to the case that $\mathcal{L}$ is a subspace lattice.

In this article, we investigate skew-adjoint interpolation problems in Alg$\mathcal{L}$: Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. Given vectors x and y in $\mathcal{H}$, when does there exist a skew-adjoint operator A in Alg$\mathcal{L}$ such that $Ax = y$?

We establish some notations and conventions. Let $\mathcal{H}$ be a Hilbert space. A *subspace lattice* $\mathcal{L}$ is a strongly closed lattice of orthogonal projections on $\mathcal{H}$. A *commutative subspace lattice* $\mathcal{L}$, or CSL $\mathcal{L}$ is a subspace lattice whose elements commute each other. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. Then Alg$\mathcal{L}$ denotes the algebra of bounded operators on $\mathcal{H}$ that leave invariant every projection in $\mathcal{L}$; Alg$\mathcal{L}$ is a weakly closed subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators acting on $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$. Then $< x, y >$ means the inner product of vectors x and y. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers. In this paper, we use the convention $\frac{0}{0} = 0$, when necessary.

2. Results

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice of orthogonal projections acting on $\mathcal{H}$ containing 0 and I. Let $\mathcal{M}$ be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{\mathcal{M}}$ means the closure of $\mathcal{M}$, $\mathcal{M}^{\perp}$ the orthogonal complement of $\mathcal{M}$ and $[\mathcal{M}]$ the closed subspace of $\mathcal{H}$ generated by $\mathcal{M}$.

Let x and y be vectors in $\mathcal{H}$. Let $sp(x) = \{ \alpha x \mid \alpha \in \mathbb{C} \}$ for a vector x of $\mathcal{H}$.

Lemma 2.1. *Let A be an operator in Alg$\mathcal{L}$ such that $Ax = y$ and $Af = 0$ for all f in $sp(x)^{\perp}$. Then the following are equivalent.*

(1) $y \in sp(x)$.

(2) *For all f in $sp(x)^{\perp}$, A^*f is a vector in $sp(x)^{\perp}$.*

Proof. (1) $\Rightarrow$ (2). Let f be a vector in $sp(x)^{\perp}$. Then

$$\begin{aligned} < A^*f, x > &= < f, Ax > \\ &= < f, y > = 0. \end{aligned}$$

Hence A^*f is a vector in $sp(x)^{\perp}$.

(2) $\Rightarrow$ (1). Let f be a vector in $sp(x)^{\perp}$. Then

$$\begin{aligned} < y, f > &=< Ax, f > \\ &=< x, A^*f >= 0. \end{aligned}$$

Hence $y \in sp(x)$. □

Lemma 2.2. *Let A be an operator in Alg$\mathcal{L}$ such that $Ax = y$ and $Af = 0$ for all f in $sp(x)^{\perp}$. If $A = -A^*$, then A^*f is a vector in $sp(x)^{\perp}$ for all $f \in sp(x)^{\perp}$.*

Proof. Let f be a vector in $sp(x)^{\perp}$ and $x = A^*x_1 + x_2$ for some x_2 in $\overline{range\ A^*}^{\perp}$. Then

$$\begin{aligned} < A^*f, x > &=< A^*f, A^*x_1 + x_2 > \\ &=< A^*f, A^*x_1 > + < A^*f, x_2 > \\ &=< A^*f, A^*x_1 > \\ &=< Af, Ax_1 > \\ &= 0. \end{aligned}$$

So A^*f is a vector in $sp(x)^{\perp}$. □

Theorem 2.3. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$ and let P_x be the projection onto $sp(x)$. If $P_xE = EP_x$ for each $E \in \mathcal{L}$, then the following are equivalent.*

(1) *There exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$, $Af = 0$ for all f in $sp(x)^{\perp}$ and $A = -A^*$.*

(2) $\sup\left\{\frac{\|E^{\perp}y\|}{\|E^{\perp}x\|} : E \in \mathcal{L}\right\} < \infty$, *$y \in sp(x)$ and* $< x, y >= - < y, x >$.

Proof. (1) $\Rightarrow$ (2). If we assume that (1) holds, then $\sup\left\{\frac{\|E^{\perp}y\|}{\|E^{\perp}x\|} : E \in \mathcal{L}\right\} < \infty$ by Jo & Kang [7, Theorem 2.4].

Since $A = -A^*$, $y \in sp(x)$ by Lemma 2.1 and 2.2. And

$$\begin{aligned} < x, y > &=< x, Ax > \\ &=< x, -A^*x > \\ &= - < y, x > . \end{aligned}$$

(2) $\Rightarrow$ (1). If $\sup\left\{\frac{\|E^{\perp}y\|}{\|E^{\perp}x\|} : E \in \mathcal{L}\right\} < \infty$, then there exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$ and $Af = 0$ for all f in $sp(x)^{\perp}$ by Jo & Kang [7, Theorem

2.4]. Since $< x,y >= - < y,x >$, $< x, Ax >= - < Ax, x >$. Let f be a vector in $sp(x)^{\perp}$. Then by Lemma 2.1, A^*f is a vector in $sp(x)^{\perp}$. Let $h = \alpha x + h_1$ be a vector in $\mathcal{H}$, where $h_1 \in sp(x)^{\perp}$. Then

$$\begin{aligned} < A^*f, h > &=< A^*f, \alpha x + h_1 > \\ &=< A^*f, \alpha x > + < A^*f, h_1 > \\ &=< f, Ah_1 > \\ &= 0. \end{aligned}$$

Hence $A^*f = 0$ for all f in $sp(x)^{\perp}$. So $A = -A^*$. □

Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ be vectors in $\mathcal{H}$.

Lemma 2.4. *Let A be an operator in $Alg\mathcal{L}$ such that $Ax_i = y_i (i = 1, 2, \ldots, n)$ and $Ag = 0$ for all g in $sp(x_1, \ldots, x_n)^{\perp}$. Then the following are equivalent.*

(1) *$y_k \in sp(x_1, \ldots, x_n)$ for all $k = 1, 2, \ldots, n$.*
(2) *If f is a vector in $sp(x_1, \ldots, x_n)^{\perp}$, A^*f is a vector in $sp(x_1, \ldots, x_n)^{\perp}$.*

Proof. (1) $\Rightarrow$ (2). Let f be a vector in $sp(x_1, \ldots, x_n)^{\perp}$. Then for all $k = 1, 2, \ldots, n$,

$$\begin{aligned} < A^*f, x_k > &=< f, Ax_k > \\ &=< f, y_k >= 0. \end{aligned}$$

So A^*f is a vector in $sp(x_1, \ldots, x_n)^{\perp}$.
(2) $\Rightarrow$ (1). Let f be a vector in $sp(x_1, \ldots, x_n)^{\perp}$. Then for all $k = 1, 2, \ldots, n$,

$$\begin{aligned} 0 =< A^*f, x_k > &=< f, Ax_k > \\ &=< f, y_k > . \end{aligned}$$

Hence $y_k \in sp(x_1, \ldots, x_n)$ for all $k = 1, 2, \ldots, n$. □

Lemma 2.5. *Let A be an operator in $Alg\mathcal{L}$ such that $Ax_i = y_i (i = 1, 2, \ldots, n)$, $Ag = 0$ for all g in $sp(x_1, \ldots, x_n)^{\perp}$ and $A = -A^*$. Then A^*f is a vector in $sp(x_1, \ldots, x_n)^{\perp}$ for all f in $sp(x_1, \ldots, x_n)^{\perp}$.*

Proof. Let f be a vector in $sp(x_1, \ldots, x_n)^\perp$ and $x_k = A^*x_{k,1} + x_{k,2}(k = 1, 2, \ldots, n)$ for some $x_{k,2} \in \overline{range\ A^*}^\perp$. Then for all $k = 1, 2, \ldots, n$,

$$\begin{aligned} < A^*f, x_k > &=< A^*f, A^*x_{k,1} + x_{k,2} > \\ &=< A^*f, A^*x_{k,1} > + < A^*f, x_{k,2} > \\ &=< A^*f, A^*x_{k,1} > \\ &=< Af, Ax_{k,1} > \\ &= 0. \end{aligned}$$

So A^*f is a vector in $sp(x_1, \ldots, x_n)^\perp$. □

Theorem 2.6. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$ and $x_1, \ldots, x_n, y_1, \ldots, y_n$ be vectors in $\mathcal{H}$. Let*

$$\mathcal{M} = \left\{ \sum_{i=1}^{n} \alpha_i x_i \ : \ \alpha_i \in \mathbb{C} \right\}$$

and $P_\mathcal{M}$ be the projection onto $\mathcal{M}$. If $P_\mathcal{M} E = E P_\mathcal{M}$ for each $E \in \mathcal{L}$, then the following are equivalent.

(1) *There is an operator A in Alg$\mathcal{L}$ such that $y_i = Ax_i (i = 1, 2, \ldots, n)$, $Ag = 0$ for all g in $sp(x_1, \ldots, x_n)^\perp$ and $A = -A^*$.*

(2)

$$\sup \left\{ \frac{\| \sum_{i=1}^{n} \alpha_i E^\perp y_i \|}{\| \sum_{i=1}^{n} \alpha_i E^\perp x_i \|} : \alpha_i \in \mathbb{C} \text{ and } E \in \mathcal{L} \right\} < \infty,$$

$$y_k \in sp(x_1, \ldots, x_n) \text{ and } < x_p, y_q >= - < y_p, x_q >$$

for all $k, p, q = 1, 2, \ldots, n$.

Proof. (1) $\Rightarrow$ (2). If we assume that (1) holds, then

$$\sup \left\{ \frac{\| \sum_{i=1}^{n} \alpha_i E^\perp y_i \|}{\| \sum_{i=1}^{n} \alpha_i E^\perp x_i \|} : \alpha_i \in \mathbb{C} \text{ and } E \in \mathcal{L} \right\} < \infty$$

by Jo & Kang [7, Theorem 2.5].

Let f be a vector in $sp(x_1, \ldots, x_n)^\perp$. Since $A = -A^*$,

$$\begin{aligned} < y_k, f > &=< Ax_k, f > \\ &=< -A^*x_k, f > \\ &= - < x_k, Af > \\ &= - < x_k, 0 >= 0 \text{ for } k = 1, 2, \ldots, n \end{aligned}$$

So $y_k \in sp(x_1, \ldots, x_n)$ for all $k = 1, 2, \ldots, n$. And

$$\begin{aligned} < x_p, y_q > &=< x_p, Ax_q > \\ &=< x_p, -A^* x_q > \\ &= - < y_p, x_q > \quad \text{for } p, q = 1, 2, \ldots, n. \end{aligned}$$

(2) $\Rightarrow$ (1). If

$$\sup \left\{ \frac{\| \sum_{i=1}^{n} \alpha_i E^{\perp} y_i \|}{\| \sum_{i=1}^{n} \alpha_i E^{\perp} x_i \|} : \alpha_i \in \mathbb{C} \text{ and } E \in \mathcal{L} \right\} < \infty,$$

then there exists an operator A in Alg$\mathcal{L}$ such that $Ax_i = y_i (i = 1, 2, \ldots, n)$ and $Af = 0$ for all f in $sp(x_1, \ldots, x_n)^{\perp}$ by Jo & Kang [7, Theorem 2.5]. Since $< x_p, y_q >= - < y_p, x_q >$, $< x_p, Ax_q >= - < Ax_p, x_q >$. Let f be a vector in $sp(x_1, \ldots, x_n)^{\perp}$. Then by Lemma 2.4, $A^* f$ is a vector in $sp(x_1, \ldots, x_n)^{\perp}$. Let $h = \sum_{i=1}^{n} \alpha_i x_i + h_1$ be a vector in $\mathcal{H}$, where $h_1 \in sp(x_1, \ldots, x_n)^{\perp}$. Since $y_k \in sp(x_1, \ldots, x_n)$ for all $k = 1, 2, \ldots, n$,

$$\begin{aligned} < A^* f, h > &=< A^* f, \sum_{i=1}^{n} \alpha_1 x_i + h_1 > \\ &=< f, A(\sum_{i=1}^{n} \alpha_i x_i) > + < f, Ah_1 > \\ &=< f, \sum_{i=1}^{n} \alpha_i y_i > \\ &= 0. \end{aligned}$$

Hence $A^* f = 0$ for all f in $sp(x_1, \ldots, x_n)^{\perp}$. So $A = -A^*$.

Let $\{x_n\}$ and $\{y_n\}$ be two infinite sequences of vectors in $\mathcal{H}$. With the similar proof as Lemma 2.4 and 2.5, we can get the following lemmas. □

Lemma 2.7. *Let A be an operator in Alg$\mathcal{L}$ such that $Ax_i = y_i (i = 1, 2, \ldots)$ and $Ag = 0$ for all g in $[x_1, \ldots, x_n, \ldots]^{\perp}$. Then the following are equivalent.*

(1) *$y_k \in [x_1, \ldots, x_n, \ldots]$ for all $k = 1, 2, \ldots$.*

(2) *If f is a vector in $[x_1, \ldots, x_n, \ldots]^{\perp}$, $A^* f$ is a vector in $[x_1, \ldots, x_n, \ldots]^{\perp}$.*

Lemma 2.8. *Let A be an operator in Alg$\mathcal{L}$ such that $Ax_i = y_i (i = 1, 2, \ldots)$, $Ag = 0$ for all g in $[x_1, \ldots, x_n, \ldots]^{\perp}$ and $A = -A^*$. Then $A^* f$ is a vector in $[x_1, \ldots, x_n, \ldots]^{\perp}$ for all f in $[x_1, \ldots, x_n, \ldots]^{\perp}$.*

With the similar proof as Theorem 2.6, we can get the following theorem.

Theorem 2.9. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$ and $\{x_n\}$ and $\{y_n\}$ be two infinite sequences of vectors in $\mathcal{H}$. Let*

$$\mathcal{M} = \left\{ \sum_{i=1}^{n} \alpha_i x_i \ : \ \alpha_i \in \mathbb{C}, n \in \mathbb{N} \right\}$$

and $P_{\overline{\mathcal{M}}}$ be the projection onto $\overline{\mathcal{M}}$. If $P_{\overline{\mathcal{M}}}E = EP_{\overline{\mathcal{M}}}$ for each $E \in \mathcal{L}$, then the following are equivalent.

(1) *There is an operator A in Alg$\mathcal{L}$ such that $y_i = Ax_i (i = 1, 2, \ldots)$, $Ag = 0$ for all g in $[x_1, \ldots, x_n, \ldots]^{\perp}$ and $A = -A^*$.*

(2)

$$\sup \left\{ \frac{\| \sum_{i=1}^{n} \alpha_i E^{\perp} y_i \|}{\| \sum_{i=1}^{n} \alpha_i E^{\perp} x_i \|} : n \in N, \alpha_i \in \mathbb{C} \text{ and } E \in \mathcal{L} \right\} < \infty,$$

$$y_k \in [x_1, \ldots, x_n, \ldots] \quad \text{and} \quad < x_p, y_q > = - < y_p, x_q >$$

for all $k, p, q = 1, 2, \ldots$.

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