J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 11, Number 1 (February 2004), Pages 51–61

ON A QUASI-SELF-SIMILAR MEASURE ON A SELF-SIMILAR SET ON THE WAY TO A PERTURBED CANTOR SET

IN-SOO BAEK

ABSTRACT. We find an easier formula to compute Hausdorff and packing dimensions of a subset composing a spectral class by local dimension of a self-similar measure on a self-similar Cantor set than that of Olsen. While we cannot apply this formula to computing the dimensions of a subset composing a spectral class by local dimension of a quasi-self-similar measure on a self-similar set on the way to a perturbed Cantor set, we have a set theoretical relationship between some distribution sets. Finally we compare the behaviour of a quasi-self-similar measure on a self-similar Cantor set with that on a self-similar set on the way to a perturbed Cantor set.

1. INTRODUCTION

Olsen [9] studied a formula to compute the Hausdorff and packing dimensions of the subset composing a multifractal spectral class of a self-similar set by a self-similar probability measure. He found the formula using some power equations essentially, so it is hard to find their solutions. We Baek [5] gave another method to find it using a set-theoretical relationship between a distribution set and a subset of same local dimension of a self-similar measure. We find it is more simpler than that of Olsen for the case of a self-similar Cantor set. Recently we Baek [6] also generalize Olsen's results to a perturbed Cantor set Baek [1, 2, 3, 4]. That is, we found a formula of computing the dimensions of the subset of same local dimension of a quasi-self-similar measure Baek [6] on a perturbed Cantor set. We find that the quasi self-similar measure in this paper plays a self-similar measure before its limit level. That is at the n-th level stage to construct a perturbed Cantor set, the n-th adjusted quasi-self-similar measure behaves like a self-similar measure on a self-similar set having 2^n contraction ratios. We need a generalized quasi-expansion

Received by the editors July 29, 2003 and, in revised form, January 27, 2004.
2000 *Mathematics Subject Classification.* 28A78.
Key words and phrases. Hausdorff dimension, packing dimension, Cantor set, distribution set.

of a point in the self-similar set to develope our theories which also need a strong law of large numbers. We naturally expected our easy computing method can be applied to that of a perturbed Cantor set, but in failure. However, we get many interesting facts of some relationship between quasi-distribution sets and generalized distribution sets (*cf.* Lee & Baek [8]).

2. PRELIMINARIES

We recall the definition of a perturbed Cantor set Baek [1]. Let $X_\phi = [0,1]$. We obtain the left subinterval $X_{\mathrm{i},1}$ and the right subinterval $X_{\mathrm{i},2}$ of X_i by deleting a middle open subinterval of X_i inductively for each $\mathrm{i} \in \{1,2\}^n$, where $n = 0,1,2,\ldots$. Let $E_n = \cup_{\mathrm{i}\in\{1,2\}^n} X_\mathrm{i}$. Then E_n is a decreasing sequence of closed sets. For each n, we set $|X_{\mathrm{i},1}|/|X_\mathrm{i}| = a_{n+1}$ and $|X_{\mathrm{i},2}|/|X_\mathrm{i}| = b_{n+1}$ for all $\mathrm{i} \in \{1,2\}^n$, where $|X|$ denotes the length of X. We assume that the contraction ratios a_n and b_n and gap ratios $1-(a_n+b_n)$ are uniformly bounded away from 0. We call $F = \cap_{n=0}^\infty E_n$ a perturbed Cantor set Baek [1]. For $\mathrm{i} \in \{1,2\}^n$, X_i denotes a fundamental interval of the n-stage of construction of perturbed Cantor set and $X_n(x)$ denotes the fundamental interval X_i containing $x \in F$.

Let $\mathbb{R}$ be the set of all real numbers and $\mathbb{N}$ be the set of all natural numbers. We note that if $x \in F$, then there is $\sigma \in \{1,2\}^\mathbb{N}$ such that

$$\bigcap_{k=0}^{\infty} I_{\sigma|k} = \{x\} \quad (\text{Here } \sigma|k = i_1, i_2, \ldots, i_k \ \text{ where } \ \sigma = i_1, i_2, \ldots, i_k, i_{k+1}, \ldots).$$

Hereafter, we use $\sigma \in \{1,2\}^\mathbb{N}$ and $x \in F$ as the same identity freely. For $y \in \mathbb{R}$, we define *a quasi-self-similar measure* μ_y *on a perturbed Cantor set* F to be a Borel probability measure on F satisfying

$$\mu_y(X_\mathrm{i}) = \frac{|X_\mathrm{i}|^y}{\prod_{k=1}^m (a_k^y + b_k^y)}$$

for $m \in \mathbb{N}$ and $\mathrm{i} \in \{1,2\}^m$.

For $n \in \mathbb{N}$ we define *a self-similar set* F_n *with contraction ratios generated by* $\{a_k, b_k\}_{k=1}^n$ by a perturbed Cantor set with $a_{hn+k} = a_k$ and $b_{hn+k} = b_k$ where $h \in \mathbb{N}$ and $k \in \{1,2,\cdots n\}$. Clearly, F_n is a self-similar set (*on the way to a perturbed Cantor set* F) having 2^n contraction ratios

$$c_{i_1,i_2,\ldots,i_n} = d_{i_1}^{(1)} d_{i_2}^{(2)} \cdots d_{i_n}^{(n)} \quad \text{where} \quad d_{i_k}^{(k)} = \begin{cases} a_k & \text{for } i_k = 1 \\ b_k & \text{for } i_k = 2 \end{cases}.$$

From now on, we write $P_n(y) = (p_1, \ldots, p_n)$ where $p_k = \frac{a_k^y}{a_k^y + b_k^y}$ and $1 \le k \le n$. We define *an n-th adjusted quasi-self-similar measure μ_y on a perturbed Cantor set F* to be the measure μ_y on the perturbed Cantor set F_n. Clearly, μ_y on F_n is a self-similar measure on F_n satisfying

$$\mu_y(X_{\mathfrak{i}}) = r_{i_1}^{(1)} r_{i_2}^{(2)} \cdots r_{i_n}^{(n)} \quad \text{where} \quad r_{i_k}^{(k)} = \begin{cases} p_k & \text{for } i_k = 1 \\ 1 - p_k & \text{for } i_k = 2 \end{cases},$$

$\mathfrak{i} = i_1, \ldots, i_k, \ldots, i_n$ and $1 \le k \le n$.

We write $E_\alpha^{P_n(y)}$ for the set of points at which the local dimension of μ_y on F_n is exactly α, so that

$$E_\alpha^{P_n(y)} = \left\{x : \lim_{r \to 0} \frac{\log \mu_y(B_r(x))}{\log r} = \alpha\right\},$$

where $B_r(x)$ is a closed ball with center x and a positive radius r. We write the above μ_y on F_n as $\gamma_{P_n(y)}$ from now on and note that $\gamma_{P_n(y)}$ is a self-similar measure on a self-similar set F_n.

Clearly, we see that a self-similar measure μ on a self-similar Cantor set (*that is,* $F_n = F_1$) satisfying $\mu(X_1) = p$ is γ_p.

We write $\underline{E}_\alpha^{(p)}$ ($\overline{E}_\alpha^{(p)}$) for the set of points at which the lower (upper) local dimension of γ_p on a self-similar Cantor set F is exactly α, so that

$$\underline{E}_\alpha^{(p)} = \left\{x : \liminf_{r \to 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \alpha\right\},$$

$$\overline{E}_\alpha^{(p)} = \left\{x : \limsup_{r \to 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \alpha\right\}.$$

In particular, we write $E_\alpha^{(p)}$ for the set of points at which the local dimension of γ_p on F is exactly α, so that

$$E_\alpha^{(p)} = \underline{E}_\alpha^{(p)} \cap \overline{E}_\alpha^{(p)}.$$

If $0 < p < 1$, then there is $y \in \mathbb{R}$ such that $P_1(y) = p$. So we note that $E_\alpha^{(p)} = E_\alpha^{P_1(y)}$. To get informations of the dimensions of $E_\alpha^{P_n(y)}$ we need the following Proposition. We write the Hausdorff dimension of a set $E \subset \mathbb{R}$ as $\dim_H(E)$ and its packing dimension as $\dim_p(E)$. The *lower* and *upper local dimension* of μ at $x \in \mathbb{R}$ are defined Falconer [7] by

$$\underline{\dim}_{loc}\mu(x) = \liminf_{r \to 0} \frac{\log \mu\big(B(x,r)\big)}{\log r},$$

$$\overline{\dim}_{loc}\mu(x) = \limsup_{r \to 0} \frac{\log \mu\big(B(x,r)\big)}{\log r}.$$

Proposition 1 (Falconer [7]). *Let $E \subset \mathbb{R}$ be a Borel set and let μ be a finite measure.*

(a) *If $\underline{\dim}_{loc}\mu(x) \geq s$ for all $x \in E$ and $\mu(E) > 0$, then $\dim_H(E) \geq s$.*

(b) *If $\underline{\dim}_{loc}\mu(x) \leq s$ for all $x \in E$, then $\dim_H(E) \leq s$.*

(c) *If $\overline{\dim}_{loc}\mu(x) \geq s$ for all $x \in E$ and $\mu(E) > 0$, then $\dim_p(E) \geq s$.*

(d) *If $\overline{\dim}_{loc}\mu(x) \leq s$ for all $x \in E$, then $\dim_p(E) \leq s$.*

Remark 1. If $A \subset E_\alpha^{P_n(y)}$ and $\gamma_{P_n(y)}(A) > 0$, then $\dim_H(A) = \dim_p(A) = \alpha$ from the above Proposition.

Lemma 2. *Let μ be a finite measure on a perturbed Cantor set F or F_n. Then for any $\alpha \geq 0$,*

$$\lim_{r\to 0} \frac{\log \mu(B_r(x))}{\log r} = \alpha \quad if\ and\ only\ if \quad \lim_{m\to\infty} \frac{\log \mu(X_m(x))}{\log |X_m(x)|} = \alpha.$$

Proof. It is obvious from the fact that the contraction ratios are uniformly bounded away from 0. □

In this paper, we assume that $0 \log 0 = 0$ for convenience.

3. MAIN RESULTS

In this section we only consider subsets in F_n.

Remark 2. Let $y \in \mathbb{R}$ and $\alpha \geq 0$. Fix $n \in \mathbb{N}$. Put $P_n(y) = (p_1, \ldots, p_n)$ where

$$p_k = \frac{a_k^y}{a_k^y + b_k^y} \quad \text{and} \quad 1 \leq k \leq n.$$

With respect to $r_1, \ldots, r_n$ we can solve the equation

$$\alpha = \frac{\sum_{k=1}^n \big(r_k \log p_k + (1 - r_k)\log(1 - p_k)\big)}{\sum_{k=1}^n \big(r_k \log a_k + (1 - r_k)\log b_k\big)} \equiv g\big(r_1, \ldots, r_n, P_n(y)\big)$$

where

$$p_k = \frac{a_k^y}{a_k^y + b_k^y}.$$

Then there exists $z \in [-\infty, \infty]$ such that $P_n(z) = (r_1, \ldots, r_n)$ and $(r_1, \ldots, r_n)$ is a solution of the above equation $\alpha = g\big(r_1, \ldots, r_n, P_n(y)\big)$. Since

$$\dim_H \left(E_\alpha^{P_n(y)}\right) = g\big(P_n(z), P_n(z)\big)$$

holds Baek [5] for $n = 1$, we naturally expect that it holds also for $n \geq 2$. In this case, we consider a self-similar measure $\gamma_{P_n(y)}$ generated by $P_n(y)$ on a self-similar set F_n with contraction ratios generated by $\{a_k, b_k\}_{k=1}^n$. Later, we see that it is a wrong conjecture.

Lemma 3. *Let*

$$G\big(P_n(z), P_n(y)\big) = \frac{\sum_{\tau\in\{1,2\}^n} r_\tau \log p_\tau}{\sum_{\tau\in\{1,2\}^n} r_\tau \log c_\tau} \quad \textit{with}$$

$$r_{i_1,i_2,\dots,i_n} = s_{i_1}^{(1)} s_{i_2}^{(2)} \dots s_{i_n}^{(n)} \qquad \textit{where} \quad s_{i_k}^{(k)} = \begin{cases} r_k & \textit{for } i_k = 1 \\ 1 - r_k & \textit{for } i_k = 2 \end{cases},$$

$$p_{i_1,i_2,\dots,i_n} = q_{i_1}^{(1)} q_{i_2}^{(2)} \cdots q_{i_n}^{(n)} \qquad \textit{where} \quad q_{i_k}^{(k)} = \begin{cases} p_k & \textit{for } i_k = 1 \\ 1 - p_k & \textit{for } i_k = 2 \end{cases},$$

$$\textit{and} \quad c_{i_1,i_2,\dots,i_n} = d_{i_1}^{(1)} d_{i_2}^{(2)} \dots d_{i_n}^{(n)} \qquad \textit{where} \quad d_{i_k}^{(k)} = \begin{cases} a_k & \textit{for } i_k = 1 \\ b_k & \textit{for } i_k = 2 \end{cases},$$

then $G\big(P_n(z), P_n(y)\big) = g\big(P_n(z), P_n(y)\big)$.

Proof. It is immediate from the cancelation. □

If $x = \sigma \in \{1,2\}^{\mathbb{N}}$, then we can express x or σ as for $x_{k,j} \in \{1,2\}$

$$x = \big((x_{1,1}, x_{2,1}, \dots, x_{k,1}, \dots, x_{n,1}), (x_{1,2}, x_{2,2}, \dots, x_{k,2}, \dots, x_{n,2}), \dots\big) \in F_n,$$

which we call *a quasi-generalized expansion of* x *in* F_n. We denote by $n_{i_1,i_2,\dots,i_n}(x|m)$ the number of times the n-tuple $(i_1, i_2, \dots, i_n)$ occurs in the first m places of the quasi-generalized expansion of

$$x = \big((x_{1,1}, x_{2,1}, \dots, x_{k,1}, \dots, x_{n,1}), (x_{1,2}, x_{2,2}, \dots, x_{k,2}, \dots, x_{n,2}), \dots\big) \in F_n.$$

For each $\mathfrak{i} \in \{1,2\}^n$ and $s_{\mathfrak{i}} \in [0,1]$ we define *a generalized distribution set* $F_n(\{s_{\mathfrak{i}}\}_{\mathfrak{i}\in\{1,2\}^n})$ *containing the finite code* $\mathfrak{i}$ *in proportion* $\{s_{\mathfrak{i}}\}_{\mathfrak{i}\in\{1,2\}^n}$ by

$$F_n(\{s_{\mathfrak{i}}\}_{\mathfrak{i}\in\{1,2\}^n}) = \Big\{x \in F_n : \lim_{m\to\infty} \frac{n_{\mathfrak{i}}(x|m)}{m} = s_{\mathfrak{i}} \ \text{ for each } \ \mathfrak{i} \in \{1,2\}^n\Big\}.$$

We denote by $n_1(x_k|m)$ the number of times the digit 1 occurs in the first k,m places of the quasi-generalized expansion of

$$x = \big((x_{1,1}, x_{2,1}, \dots, x_{k,1}, \dots, x_{n,1}), (x_{1,2}, x_{2,2}, \dots, x_{k,2}, \dots, x_{n,2}), \cdots\big) \in F_n.$$

For $(r_1, \dots, r_n) \in [0,1]^n$, we define *a quasi-distribution set* $F_n^*(r_1, \dots, r_n)$ *containing the digit 1 in proportion* $(r_1, \dots, r_n)$ by

$$F_n^*(r_1, \dots, r_n) = \{x \in F_n : \lim_{m\to\infty} \frac{n_1(x_k|m)}{m} = r_k \text{ for each } \ 1 \le k \le n\}.$$

Lemma 4. *For* $\mathfrak{i} = i_1, i_2, \ldots, i_n$ *and*

$$s_{\mathfrak{i}} = s_{i_1}^{(1)} s_{i_2}^{(2)} \cdots s_{i_n}^{(n)} \quad \textit{where} \quad s_{i_k}^{(k)} = \begin{cases} r_k & \textit{for } i_k = 1 \\ 1 - r_k & \textit{for } i_k = 2 \end{cases},$$

$$\{x \in F_n : \lim_{m\to\infty} \frac{n_{\mathfrak{i}}(x|m)}{m} = s_{\mathfrak{i}} \ \textit{for each} \ \mathfrak{i} \in \{1,2\}^n\}$$
$$\subset \ \{x \in F_n : \lim_{m\to\infty} \frac{n_1(x_k|m)}{m} = r_k \ \textit{for each} \ 1 \le k \le n\}.$$

Proof. For $m \in \mathbb{N}$,

$$\frac{n_1(x_k|m)}{m} = \sum_{i_k=1} \frac{n_{i_1,i_2,\ldots,i_k,\ldots,i_n}(x|m)}{m}.$$

We easily obtain it from the limit of each term. □

Remark 3. In the above Proof, for $n = 3$,

$$\frac{n_1(x_1|m)}{m} = \frac{n_{111}(x|m)}{m} + \frac{n_{112}(x|m)}{m} + \frac{n_{121}(x|m)}{m} + \frac{n_{122}(x|m)}{m},$$
$$\frac{n_1(x_2|m)}{m} = \frac{n_{111}(x|m)}{m} + \frac{n_{112}(x|m)}{m} + \frac{n_{211}(x|m)}{m} + \frac{n_{212}(x|m)}{m},$$
$$\frac{n_1(x_3|m)}{m} = \frac{n_{111}(x|m)}{m} + \frac{n_{121}(x|m)}{m} + \frac{n_{211}(x|m)}{m} + \frac{n_{221}(x|m)}{m}.$$

Remark 4. Since from the strong law of large numbers (*cf.* Lee & Baek [8])

$$\gamma_{\{s_{\mathfrak{i}}\}_{\mathfrak{i}\in\{1,2\}^n}}\big(\{x \in F_n : \lim_{m\to\infty} \frac{n_{\mathfrak{i}}(x|m)}{m} = s_{\mathfrak{i}} \ \text{for each} \ \mathfrak{i} \in \{1,2\}^n\}\big) = 1,$$

we see that

$$\gamma_{\{s_{\mathfrak{i}}\}_{\mathfrak{i}\in\{1,2\}^n}}\big(\{x \in F_n : \lim_{m\to\infty} \frac{n_1(x_k|m)}{m} = r_k \ \text{for each} \ 1 \le k \le n\}\big) = 1.$$

By the notation in the Preliminaries, we see that a self-similar measure $\gamma_{\{s_{\mathfrak{i}}\}_{\mathfrak{i}\in\{1,2\}^n}}$ in the above is $\gamma_{P_n(z)}$ where $P_n(z) = (r_1, \ldots, r_n)$. From now on, we write a generalized distribution set

$$\{x \in F_n : \lim_{m\to\infty} \frac{n_{\mathfrak{i}}(x|m)}{m} = s_{\mathfrak{i}} \ \text{for each} \ \mathfrak{i} \in \{1,2\}^n\}$$

containing the finite codes $\mathfrak{i}$ in proportion $s_{\mathfrak{i}}$ in the above Lemma as $F_n(P_n(z))$.

Theorem 5. *Fix* $n \in \mathbb{N}$ *and consider a self-similar set* F_n *with contraction ratios generated by* $\{a_k, b_k\}_{k=1}^n$. *Let* $y \in (-\infty, \infty)$ *and consider a self-similar measure* $\gamma_{P_n(y)}$ *on* F_n *where* $P_n(y) = (p_1, \ldots, p_n)$ *and* $p_k = \frac{a_k^y}{a_k^y + b_k^y}$ *for* $1 \le k \le n$. *Let* $z \in [-\infty, \infty]$ *and consider*

$$g(P_n(z), P_n(y)) = \frac{\sum_{k=1}^n \big(r_k \log p_k + (1 - r_k)\log(1 - p_k)\big)}{\sum_{k=1}^n \big(r_k \log a_k + (1 - r_k)\log b_k\big)}$$

where $P_n(z) = (r_1, \ldots, r_n)$ *and* $r_k = \frac{a_k^z}{a_k^z + b_k^z}$ *for* $1 \le k \le n$. *Then*

$$F_n^*\big(P_n(z)\big) \subset E^{P_n(y)}_{g\left(P_n(z), P_n(y)\right)}.$$

Proof. Let $x \in F_n^*(P_n(z))$. Then

$$\begin{aligned}
&\lim_{m\to\infty} \frac{\log \gamma_{P_n(y)}(c_m(x))}{\log |c_m(x)|} \\
=\ & \lim_{m\to\infty} \frac{\sum_{k=1}^n \big(n_1(x_k|m)\log p_k + \big(m - n_1(x_k|m)\log(1-p_k)\big)}{\sum_{k=1}^n \big(n_1(x_k|m)\log a_k + (m - n_1(x_k|m)\log b_k)\big)} \\
=\ & \frac{\sum_{k=1}^n \big(r_k \log p_k + (1-r_k)\log(1-p_k)\big)}{\sum_{k=1}^n \big(r_k \log a_k + (1-r_k)\log b_k\big)} \\
=\ & g\big(P_n(z), P_n(y)\big).
\end{aligned}$$

□

Corollary 6. $F_n(P_n(z)) \subset F_n^*(P_n(z)) \subset E^{P_n(z)}_{g(P_n(z),P_n(z))}$ *where* $z \in \mathbb{R}$, *and* $F_n(P_n(z)) \subset F_n^*(P_n(z)) \subset E^{P_n(y)}_{g(P_n(z),P_n(y))}$ *where* $z \in [-\infty, \infty]$ *and* $y \in \mathbb{R}$.

Proof. It is immediate from Lemma 4 and the above Theorem. □

Remark 5. From now on, we will not designate the ranges of z and y if there is no confusion. That is, if we consider $E^{P_n(z)}_{g(P_n(z),P_n(z))}$ then $z \in \mathbb{R}$ and if we consider $E^{P_n(y)}_{g(P_n(z),P_n(y))}$ then $y \in \mathbb{R}$ and $z \in [-\infty, \infty]$.

Theorem 7. $\gamma_{P_n(z)}\big(F_n^*(P_n(z))\big) = \gamma_{P_n(z)}\big(E^{P_n(z)}_{g(P_n(z),P_n(z))}\big) = 1$. *Further,*

$$\dim_H\big(F_n^*(P_n(z))\big) = g\big(P_n(z), P_n(z)\big) = \dim_p\big(F_n^*(P_n(z))\big) \text{ and}$$
$$\dim_H\big(E^{P_n(z)}_{g(P_n(z),P_n(z))}\big) = g(P_n(z), P_n(z)) = \dim_p\big(E^{P_n(z)}_{g(P_n(z),P_n(z))}\big).$$

Proof. It follows from the above Remark. That is, $\gamma_{P_n(z)}\big(F_n^*(P_n(z))\big) = 1$ follows from $F_n(P_n(z)) \subset F_n^*(P_n(z))$ and $\gamma_{P_n(z)}\big(F_n(P_n(z))\big) = 1$ from the strong law of large numbers. Further,

$$\dim_H\big(F_n^*(P_n(z))\big) = g\big(P_n(z), P_n(z)\big) = \dim_p\big(F_n^*(P_n(z))\big)$$

follows from the above Corollary and Remark 1 in the Preliminaries. Similarly, by Proposition 1, we have

$$\dim_H\big(E^{P_n(z)}_{g(P_n(z),P_n(z))}\big) = g\big(P_n(z), P_n(z)\big) = \dim_p\big(E^{P_n(z)}_{g(P_n(z),P_n(z))}\big).$$

□

Remark 6.

$$F_n(P_n(z)) = F_n^*(P_n(z)) = E_{g(P_n(z),P_n(y))}^{P_n(y)}$$

for $n = 1$ (*cf.* Baek [5]). So

$$\dim_H \left(E_{g(P_n(z),P_n(y))}^{P_n(y)}\right) = g\big(P_n(z), P_n(z)\big) = \dim_p \left(E_{g(P_n(z),P_n(y))}^{P_n(y)}\right)$$

for $n = 1$. However, from the above Corollary and Theorem, we just find that $g\big(P_n(z), P_n(z)\big)$ is a lower bound for the dimensions of $E_{g(P_n(z),P_n(y))}^{P_n(y)}$.

Theorem 8. *If s is a real number satisfying*

$$\prod_{k=1}^{n}(a_k^s + b_k^s) = 1,$$

then $g\big(P_n(s), P_n(s)\big) = s$. Further, $E_s^{P_n(s)} = F_n$ and $\dim_H(F_n) = \dim_p(F_n) = s$.

Proof. Put

$$r_k = \frac{a_k^s}{a_k^s + b_k^s} \text{ in } g(r_1, \ldots, r_n, r_1, \ldots, r_n).$$

Then we easily see that

$$g\big(P_n(s), P_n(s)\big) = \frac{s\big(\sum_{k=1}^{n}(r_k \log a_k + (1 - r_k)\log b_k)\big) - \sum_{k=1}^{n}\log(a_k^s + b_k^s)}{\sum_{k=1}^{n}\big(r_k \log a_k + (1 - r_k)\log b_k\big)} = s.$$

Further, by Lemma 2 in the Preliminaries we easily see that $E_s^{P_n(s)} = F_n$ and

$$\dim_H \left(E_{g(P_n(s),P_n(s))}^{P_n(s)}\right) = g\big(P_n(s), P_n(s)\big) = \dim_p \left(E_{g(P_n(s),P_n(s))}^{P_n(s)}\right).$$

□

Proposition 9. *Let*

$$H(P_n(y)) = \frac{\sum_{\tau\in\{1,2\}^n} r_\tau \log p_\tau}{\sum_{\tau\in\{1,2\}^n} r_\tau \log c_\tau} \quad \textit{with}$$

$$r_{i_1,\ldots,i_n} = s_{i_1}^{(1)} \cdots s_{i_n}^{(n)} \qquad \textit{where} \quad s_{i_k}^{(k)} = \begin{cases} {p_k}^q {a_k}^{\beta(q)} & \textit{for } i_k = 1 \\ (1-p_k)^q {b_k}^{\beta(q)} & \textit{for } i_k = 2 \end{cases},$$

$$p_{i_1,i_2,\cdots,i_n} = q_{i_1}^{(1)} q_{i_2}^{(2)} \cdots q_{i_n}^{(n)} \quad \textit{where} \quad q_{i_k}^{(k)} = \begin{cases} p_k & \textit{for } i_k = 1 \\ 1 - p_k & \textit{for } i_k = 2 \end{cases},$$

$$\textit{and} \quad c_{i_1,i_2,\ldots,i_n} = d_{i_1}^{(1)} d_{i_2}^{(2)} \ldots d_{i_n}^{(n)} \quad \textit{where} \quad d_{i_k}^{(k)} = \begin{cases} a_k & \textit{for } i_k = 1 \\ b_k & \textit{for } i_k = 2 \end{cases}.$$

Then the solution q satisfying

$$H\big(P_n(y)\big) = \alpha \quad \text{and} \quad \prod_{k=1}^{n} \left({p_k}^q {a_k}^{\beta(q)} + (1-p_k)^q {b_k}^{\beta(q)}\right) = 1$$

gives $\alpha q + \beta(q)$ as the dimensions of $E_\alpha^{P_n(y)} (\subset F_n)$.

Proof. It is immediate from (11.30) and (11.35) in Falconer [7]. □

Remark 7. In the above Proposition, if $q = 1$ then $\beta(q) = 0$ in the equation

$$\prod_{k=1}^{n} \left({p_k}^q {a_k}^{\beta(q)} + (1-p_k)^q {b_k}^{\beta(q)}\right) = 1.$$

Further, for $q = 1$ let $\alpha = H\big(P_n(y)\big)$. Then by the above Proposition the dimensions of $E_\alpha^{P_n(y)}$ are α. By the way, $\alpha = H\big(P_n(y)\big) = G\big(P_n(y), P_n(y)\big) = g\big(P_n(y), P_n(y)\big)$ from Lemma 3. By the Theorem 7, we also see that the dimensions of $E_\alpha^{P_n(y)}$ are $g\big(P_n(y), P_n(y)\big) = \alpha$.

Theorem 10.

For $n = 1, E^{P_n(y)}_{g(P_n(z),P_n(y))} = E^{P_n(z)}_{g(P_n(z),P_n(z))}$.

For $n \geq 2$, *in general,* $E^{P_n(y)}_{g(P_n(z),P_n(y))} \neq E^{P_n(z)}_{g(P_n(z),P_n(z))}$.

Further, $\dim_H \left(E^{P_n(y)}_{g(P_n(z),P_n(y))}\right) = \dim_p \left(E^{P_n(y)}_{g(P_n(z),P_n(y))}\right) \geq g\big(P_n(z), P_n(z)\big)$.

Proof. For $n = 1$, it follows from Baek [5]. For $n \geq 2$, it is immediate from the above Proposition and Lemma 3.

$$\dim_H \left(E^{P_n(y)}_{g(P_n(z),P_n(y))}\right) = \dim_p \left(E^{P_n(y)}_{g(P_n(z),P_n(y))}\right) \geq g\big(P_n(z), P_n(z)\big)$$

follows from Remark 6. □

Remark 8. In the above Proof, for $n \geq 2$ we cannot guarantee that

$${p_k}^q {a_k}^{\beta(q)} + (1-p_k)^q {b_k}^{\beta(q)} = 1$$

for each $1 \leq k \leq n$ in the above Proposition whereas $r_k + (1 - r_k) = 1$ for each $1 \leq k \leq n$ in Lemma 3. However, if we guarantee it,

$$\alpha = H\big(P_n(y)\big) = G\big(P_n(z), P_n(y)\big) = g\big(P_n(z), P_n(y)\big)$$

where $P_n(z) = \{{p_k}^q {a_k}^{\beta(q)}\}_{k=1}^n$ from Lemma 3. Then we easily see that

$$g\big(P_n(z), P_n(z)\big) = G\big(P_n(z), P_n(z)\big) = \alpha q + \beta(q),$$

which is the dimensions of $E^{P_n(y)}_{g(P_n(z),P_n(y))}$. But we know

$$\dim_H \left(E^{P_n(z)}_{g(P_n(z),P_n(z))}\right) = g\big(P_n(z), P_n(z)\big) = \dim_p \left(E^{P_n(z)}_{g(P_n(z),P_n(z))}\right)$$

from Theorem 7. This gives many examples for $E^{P_n(y)}_{g(P_n(z),P_n(y))} \neq E^{P_n(z)}_{g(P_n(z),P_n(z))}$ for $n \geq 2$. But for $n = 1$, letting

$$\begin{aligned} &{p_1}^q {a_1}^{\beta(q)} + (1-p_1)^q {b_1}^{\beta(q)} = 1 \quad \textit{and} \\ &r_1 = {p_1}^q {a_1}^{\beta(q)} \quad \textit{and} \\ &r_2 = 1 - r_1 = (1-p_1)^q {b_1}^{\beta(q)} \end{aligned}$$

in Lemma 3, we have $P_1(z) = r_1$ and $g(r_1, r_1) = \alpha q + \beta(q)$. Precisely, the solution q satisfying

$$H(P_1(y)) = H(p_1) = \alpha \quad \text{and} \quad \prod_{k=1}^{1} \left({p_k}^q {a_k}^{\beta(q)} + (1-p_k)^q {b_k}^{\beta(q)}\right) = 1$$

gives $r_1 = {p_1}^q {a_1}^{\beta(q)}$ and $g(r_1, r_1) = \alpha q + \beta(q)$. Further, we see that

$$g\big(P_n(z), P_n(y)\big) \geq g\big(P_n(z), P_n(z)\big)$$

from the Lagrange multiplier theorem. However, we also see it from the Proposition 1 and the above theorem, that is

$$g\big(P_n(z), P_n(z)\big) \leq \dim_H \left(E^{P_n(y)}_{g(P_n(z),P_n(y))}\right) \leq g\big(P_n(z), P_n(y)\big).$$

Theorem 11. *Let s be a real number satisfying*

$$\prod_{k=1}^{n} (a_k^s + b_k^s) = 1 \quad \textit{and let} \quad z \in [-\infty, \infty].$$

Then for any $y \neq y'$ in $\mathbb{R}$,

$$\textit{for} \quad n = 1, \quad E^{P_n(y)}_{g(P_n(z),P_n(y))} = E^{P_n(y')}_{g(P_n(z),P_n(y'))} \quad \textit{if} \quad y \neq s,$$

for $n \geq 2$, we cannot guarantee

$$E^{P_n(y)}_{g(P_n(z),P_n(y))} = E^{P_n(y')}_{g(P_n(z),P_n(y'))} \quad \textit{if} \quad y \neq s.$$

Proof. It is immediate from the above Theorem and Baek [5]. □

REFERENCES

1. I. S. Baek: Dimensions of the perturbed Cantor set. *Real Anal. Exchange* **19** (1993/94), no. 1, 269–273. MR **95c**:28007
2. ______: Hausdorff dimension of perturbed Cantor sets without some boundedness condition. *Acta Math. Hungar.* **99** (2003), no. 4, 279–283. CMP 1981929
3. ______: Dimensions of measures on perturbed Cantor set. *J. Appl. Math. & Comput.* **14(1-2)** (2004), 397–403.
4. ______: Cantor dimension and its application. *Bull. Korean Math. Soc.* To appear.
5. ______: Relation between spectral classes of a self-similar Cantor set. *J. Math. Anal. Appl.* To appear.
6. ______: Multifractal spectra by quasi-self-similar measures on a perturbed Cantor set. submitted.
7. K. J. Falconer: Techniques in fractal geometry. *John Wiley & Sons, Ltd.*, Chichester, 1997. MR **99f**:28013
8. H. H. Lee & I. S. Baek: Dimensions of a Cantor type set and its distribution sets. *Kyungpook Math. J.* **32** (1992), no. 2, 149–152. MR **93m**:58069
9. L. Olsen: A multifractal formalism. *Adv. Math.* **116** (1995), no. 1, 82–196. MR **97a**:28006

DEPARTMENT OF MATHEMATICS, PUSAN UNIVERSITY OF FOREIGN STUDIES, 55-1 UAM 2-DONG, NAM-GU, BUSAN 608-738, KOREA
Email address: isbaek@pufs.ac.kr