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LINEAR MAPPINGS, QUADRATIC MAPPINGS AND CUBIC MAPPINGS IN NORMED SPACES

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ABSTRACT. It is shown that every almost linear mapping $h : X \to Y$ of a complex normed space X to a complex normed space Y is a linear mapping when $h(rx) = rh(x)$ $(r > 0, r \neq 1)$ holds for all $x \in X$, that every almost quadratic mapping $h : X \to Y$ of a complex normed space X to a complex normed space Y is a quadratic mapping when $h(rx) = r^2h(x)$ $(r > 0, r \neq 1)$ holds for all $x \in X$, and that every almost cubic mapping $h : X \to Y$ of a complex normed space X to a complex normed space Y is a cubic mapping when $h(rx) = r^3h(x)$ $(r > 0, r \neq 1)$ holds for all $x \in X$.

1. INTRODUCTION

Let X and Y be Banach spaces with norms $||\cdot||$ and $||\cdot||$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Rassias [10] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in X$. Găvruţa [2] generalized the Rassias' result: Let G be an abelian group and Y a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y) < \infty$$

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for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in G$. Park [9] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra.

A mapping $f : X \to Y$ is called $\mathbb{C}$-*quadratic* if f satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad \text{and} \quad f(\lambda x) = \lambda^2 f(x)$$

for all $x, y \in X$. Skof [11] was the first author to treat the Hyers-Ulam stability of a quadratic functional equation. Czerwik [1] generalized the Skof's result.

A mapping $f : X \to Y$ is called $\mathbb{C}$-*cubic* if f satisfies the functional equation

$$f(x+2y) + f(x-2y) + 6f(x) = 4f(x+y) + 4f(x-y) \quad \text{and} \quad f(\lambda x) = \lambda^3 f(x)$$

for all $x, y \in X$. Jun & Kim [4] were the authors to treat the Hyers-Ulam stability of a cubic functional equation.

The stability problems of functional equations have been investigated by several authors [3, 5, 6, 7].

Throughout this paper, let X and Y be complex normed spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively, and r $(r \neq 1)$ a positive real number.

Using the stability methods, we prove that every almost linear mapping $h : X \to Y$ is a linear mapping when $h(rx) = rh(x)$ holds for all $x \in X$, that every almost quadratic mapping $h : X \to Y$ is a quadratic mapping when $h(rx) = r^2 h(x)$ holds for all $x \in X$, and that every almost cubic mapping $h : X \to Y$ is a cubic mapping when $h(rx) = r^3 h(x)$ holds for all $x \in X$.

2. Linear mappings, quadratic mappings, and cubic mappings in complex normed spaces

We are going to investigate linear mappings in complex normed spaces.

Theorem 1. *Let $h : X \to Y$ be a mapping satisfying $h(rx) = rh(x)$ for all $x \in X$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ such that*

(i) $\sum_{j=0}^{\infty} \frac{1}{r^j}\varphi(r^j x, r^j y) < \infty$ *for $x, y \in X$, and*

(ii) $\|h(\mu x+\mu y)-\mu h(x)-\mu h(y)\| \le \varphi(x,y)$ *for all* $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ *and all* $x, y \in X$.

Then the mapping $h : X \to Y$ *is a* $\mathbb{C}$*-linear mapping.*

Proof. Since $h(0) = rh(0)$, $h(0) = 0$. Put $\mu = 1 \in \mathbb{T}^1$ in (ii). By (ii) and the assumption that $h(rx) = rh(x)$ for all $x \in X$,

$$\|h(x+y)-h(x)-h(y)\| = \frac{1}{r^n}\|h(r^n x + r^n y) - h(r^n x) - h(r^n y)\| \le \frac{1}{r^n}\varphi(r^n x, r^n y),$$

which tends to zero as $n \to \infty$ by (i). So

$$h(x+y) = h(x) + h(y) \text{ for all } x, y \in X.$$

Put $y = 0$ in (ii). By (ii) and the assumption that $h(rx) = rh(x)$ for all $x \in X$,

$$\|h(\mu x) - \mu h(x)\| = \frac{1}{r^n}\|h(r^n \mu x) - \mu h(r^n x)\| \le \frac{1}{r^n}\varphi(r^n x, 0),$$

which tends to zero as $n \to \infty$ by (i). So

$$h(\mu x) = \mu h(x) \text{ for all } \mu \in \mathbb{T}^1 \text{ and all } x \in X. \tag{1}$$

Now let $\lambda \in \mathbb{C}$ ($\lambda \ne 0$) and M an integer greater than $4|\lambda|$. Then

$$\left|\frac{\lambda}{M}\right| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By Kadison & Pedersen [8, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. So by (1)

$$\begin{aligned}
h(\lambda x) &= h\left(\frac{M}{3} \cdot 3\frac{\lambda}{M}x\right) \\
&= M \cdot h\left(\frac{1}{3} \cdot 3\frac{\lambda}{M}x\right) \\
&= \frac{M}{3}h\left(3\frac{\lambda}{M}x\right) \\
&= \frac{M}{3}h(\mu_1 x + \mu_2 x + \mu_3 x) \\
&= \frac{M}{3}(h(\mu_1 x) + h(\mu_2 x) + h(\mu_3 x)) \\
&= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)h(x) \\
&= \frac{M}{3} \cdot 3\frac{\lambda}{M}h(x) \\
&= \lambda h(x)
\end{aligned}$$

for all $x \in X$. Hence

$$h(\zeta x + \eta y) = h(\zeta x) + h(\eta y) = \zeta h(x) + \eta h(y)$$

for all $\zeta, \eta \in \mathbb{C}(\zeta, \eta \neq 0)$ and all $x, y \in X$. And $h(0x) = 0 = 0h(x)$ for all $x \in X$. So the mapping $h : X \to Y$ is a $\mathbb{C}$-linear mapping, as desired. □

Corollary 2. *Let $r > 1$. Let $h : X \to Y$ be a mapping satisfying $h(rx) = rh(x)$ for all $x \in X$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in X$. Then the mapping $h : X \to Y$ is a $\mathbb{C}$-linear mapping.

Proof. Define $\varphi(x, y) = \theta(||x||^p + ||y||^p)$, and apply Theorem 1. □

Corollary 3. *Let $0 < r < 1$. Let $h : X \to Y$ be a mapping satisfying $h(rx) = rh(x)$ for all $x \in X$ for which there exist constants $\theta \geq 0$ and $p \in (1, \infty)$ such that*

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in X$. Then the mapping $h : X \to Y$ is a $\mathbb{C}$-linear mapping.

Proof. Define $\varphi(x, y) = \theta(||x||^p + ||y||^p)$, and apply Theorem 1. □

Theorem 4. *Let $h : X \to Y$ be a mapping satisfying $h(rx) = rh(x)$ for all $x \in X$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying* (i) *such that*

(iii) $\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \leq \varphi(x, y)$

for $\mu = 1$, i, and all $x, y \in X$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping $h : X \to Y$ is a $\mathbb{C}$-linear mapping.

Proof. Put $\mu = 1$ in (iii). By the same reasoning as the proof of Theorem 1, the mapping $h : X \to Y$ is additive. By the same reasoning as the proof of Rassias [10, Theorem], the additive mapping $h : X \to Y$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (iii). By the same method as the proof of Theorem 1, one can obtain that

$$h(ix) = ih(x)$$

for all $x \in X$. For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$h(\lambda x) = h(sx + itx) = sh(x) + th(ix) = sh(x) + ith(x) = (s + it)h(x) = \lambda h(x)$$

for all $\lambda \in \mathbb{C}$ and all $x \in X$. So

$$h(\zeta x + \eta y) = h(\zeta x) + h(\eta y) = \zeta h(x) + \eta h(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in X$. Hence the mapping $h : X \to Y$ is $\mathbb{C}$-linear, as desired. □

Now we are going to investigate quadratic mappings in complex normed spaces.

Theorem 5. *Let $h : X \to Y$ be a mapping satisfying $h(rx) = r^2 h(x)$ for all $x \in X$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ such that*

(iv) $\sum_{j=0}^{\infty} \frac{1}{r^{2j}} \varphi(r^j x, r^j y) < \infty$,

(v) $\|h(\lambda x + \lambda y) + h(\lambda x - \lambda y) - 2\lambda^2 h(x) - 2\lambda^2 h(y)\| \leq \varphi(x, y)$

for all $\lambda \in \mathbb{C}$ and all $x, y \in X$. Then the mapping $h : X \to Y$ is a $\mathbb{C}$-quadratic mapping.

Proof. Since $h(0) = r^2 h(0)$, $h(0) = 0$. Put $\lambda = 1$ in (v). By (v) and the assumption that $h(rx) = r^2 h(x)$ for all $x \in X$,

$$\begin{aligned} &\|h(x+y) + h(x-y) - 2h(x) - 2h(y)\| \\ &= \frac{1}{r^{2n}} \|h(r^n x + r^n y) + h(r^n x - r^n y) - 2h(r^n x) - 2h(r^n y)\| \leq \frac{1}{r^{2n}} \varphi(r^n x, r^n y), \end{aligned}$$

which tends to zero as $n \to \infty$ by (iv). So

$$h(x+y) + h(x-y) = 2h(x) + 2h(y)$$

for all $x, y \in X$.

Put $y = 0$ in (v). By (v) and the assumption that $h(rx) = r^2 h(x)$ for all $x \in X$,

$$\|2h(\lambda x) - 2\lambda^2 h(x)\| = \frac{1}{r^{2n}} \|2h(r^n \lambda x) - 2\lambda^2 h(r^n x)\| \leq \frac{1}{r^{2n}} \varphi(r^n x, 0),$$

which tends to zero as $n \to \infty$ by (iv). So

$$h(\lambda x) = \lambda^2 h(x)$$

for all $\lambda \in \mathbb{C}$ and all $x \in X$. So the mapping $h : X \to Y$ is a $\mathbb{C}$-quadratic mapping, as desired. □

Corollary 6. *Let $r > 1$. Let $h : X \to Y$ be a mapping satisfying $h(rx) = r^2 h(x)$ for all $x \in X$ for which there exist constants $\theta \geq 0$ and $p \in [0, 2)$ such that*

$$\|h(\lambda x + \lambda y) + h(\lambda x - \lambda y) - 2\lambda^2 h(x) - 2\lambda^2 h(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $\lambda \in \mathbb{C}$ and all $x, y \in X$. Then the mapping $h : X \to Y$ is a $\mathbb{C}$-quadratic mapping.

Proof. Define $\varphi(x, y) = \theta(||x||^p + ||y||^p)$, and apply Theorem 5. □

Corollary 7. *Let $0 < r < 1$. Let $h : X \to Y$ be a mapping satisfying $h(rx) = r^2 h(x)$ for all $x \in X$ for which there exist constants $\theta \geq 0$ and $p \in (2, \infty)$ such that*

$$\|h(\lambda x + \lambda y) + h(\lambda x - \lambda y) - 2\lambda^2 h(x) - 2\lambda^2 h(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $\lambda \in \mathbb{C}$ and all $x, y \in X$. Then the mapping $h : X \to Y$ is a $\mathbb{C}$-quadratic mapping.

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 5. □

We are going to investigate cubic mappings in complex normed spaces.

Theorem 8. *Let $h : X \to Y$ be a mapping satisfying $h(rx) = r^3 h(x)$ for all $x \in X$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ such that*

(vi) $\sum_{j=0}^{\infty} \frac{1}{r^{3j}} \varphi(r^j x, r^j y) < \infty$,
(vii) $\|h(\lambda x + 2\lambda y) + h(\lambda x - 2\lambda y) + 6h(\lambda x) - 4\lambda^3 h(x + y) - 4\lambda^3 h(x - y)\| \leq \varphi(x, y)$

for all $\lambda \in \mathbb{C}$ and all $x, y \in X$. Then the mapping $h : X \to Y$ is a $\mathbb{C}$-cubic mapping.

Proof. Since $h(0) = r^3 h(0)$, $h(0) = 0$. Put $\lambda = 1$ in (vii). By (vii) and the assumption that $h(rx) = r^3 h(x)$ for all $x \in X$,

$$\begin{aligned}
&\|h(x + 2y) + h(x - 2y) + 6h(x) - 4h(x + y) - 4h(x - y)\| \\
&= \frac{1}{r^{3n}} \|h(r^n x + 2r^n y) + h(r^n x - 2r^n y) + 6h(r^n x) - 4h(r^n x + r^n y) - 4h(r^n x - r^n y)\| \\
&\leq \frac{1}{r^{3n}} \varphi(r^n x, r^n y),
\end{aligned}$$

which tends to zero as $n \to \infty$ by (vi). So

$$h(x + 2y) + h(x - 2y) + 6h(x) = 4h(x + y) + 4h(x - y)$$

for all $x, y \in X$.

Put $y = 0$ in (vii). By (vii) and the assumption that $h(rx) = r^3 h(x)$ for all $x \in X$,

$$\|8h(\lambda x) - 8\lambda^3 h(x)\| = \frac{1}{r^{3n}} \|8h(r^n \lambda x) - 8\lambda^3 h(r^n x)\| \leq \frac{1}{r^{3n}} \varphi(r^n x, 0),$$

which tends to zero as $n \to \infty$ by (vi). So

$$h(\lambda x) = \lambda^3 h(x)$$

for all $\mu \in \mathbb{C}$ and all $x \in X$. So the mapping $h : X \to Y$ is a $\mathbb{C}$-cubic mapping, as desired. □

Corollary 9. *Let $r > 1$. Let $h : X \to Y$ be a mapping satisfying $h(rx) = r^3h(x)$ for all $x \in X$ for which there exist constants $\theta \geq 0$ and $p \in [0,3)$ such that*

$$\|h(\lambda x+2\lambda y)+h(\lambda x-2\lambda y)+6h(\lambda x)-4\lambda^3h(x+y)-4\lambda^3h(x-y)\| \leq \theta(\|x\|^p+\|y\|^p)$$

for all $\lambda \in \mathbb{C}$ and all $x, y \in X$. Then the mapping $h : X \to Y$ is a $\mathbb{C}$-cubic mapping.

Proof. Define $\varphi(x,y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 8. □

Corollary 10. *Let $0 < r < 1$. Let $h : X \to Y$ be a mapping satisfying $h(rx) = r^3h(x)$ for all $x \in X$ for which there exist constants $\theta \geq 0$ and $p \in (3,\infty)$ such that*

$$\|h(\lambda x+2\lambda y)+h(\lambda x-2\lambda y)+6h(\lambda x)-4\lambda^3h(x+y)-4\lambda^3h(x-y)\| \leq \theta(\|x\|^p+\|y\|^p)$$

for all $\lambda \in \mathbb{C}$ and all $x, y \in X$. Then the mapping $h : X \to Y$ is a $\mathbb{C}$-cubic mapping.

Proof. Define $\varphi(x,y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 8. □

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